A discretized Chern-Simons gauge theory on arbitrary graphs

Kai Sun

University of Michigan, Ann Arbor

Collaborators: Krishna Kumar and Eduardo Fradkin (UIUC)
Outline

- How to construct a discretized Chern-Simons gauge theory
  - A necessary and sufficient condition for such a theory to be simple (quadratic, local, etc.)

- Verify that the discretized theory preserves all necessary properties of the CS theory in the continuum

- Application I: the topological field theory for fractional Chern insulators
  - What is the smallest setup to support a FQH system

- Application II: Chiral spin liquids
The Chern-Simons gauge theory

\[ S = \frac{k}{4\pi} \int d\vec{r} dt \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \]

- A topological field theory
  - Quantized coefficient: \( k \) is an integer
  - Topological degeneracy on manifold with non-zero genus
- Integer and fractional quantum Hall effect
  - The response theory for the integer quantum Hall effect
  - The topological field theory for the fractional quantum Hall effect
- Chiral spin liquids

- A quadratic theory
- Written in terms of \( A \), instead of \( E \) and \( B \) (be careful about gauge invariance)
Why discretizing?

- Fractional Chern insulators (FQHE in lattice systems with B=0)

Q: What is the topological field theory description for the FCI?

- Chiral Spin liquid in frustrated spin systems
  - Spin-1/2 doesn’t follow the canonical commutation relation
  - Can be mapped into hard-core bosons (with infinite interactions)
  - Map into fermions?
    - Don’t need infinite interactions
    - But spins are bosons, not fermions
    - 1D: Fermionization (the Jordan-Wigner transformation)
    - 2D: Fermions coupled with a C-S gauge field (Fradkin 1989 and 1994)

For frustrated spin systems, the lattice structure plays a crucial role.

Objectives

- Discretizing the space (not time)
- A quadratic theory (same as the continuum)
- Doesn’t require translational symmetries: lattices and graphs (important for the QHE)
- Can be embedded on arbitrary 2D manifolds with arbitrary genus (important for the QHE)
- Shall preserve all topological properties of the CS theory in the continuum

Constraints on the lattices (graphs):
- Must be planar (no crossing bonds)
- Must be simple
  - Only 1 bond between any two neighboring sites
  - No loop with length 1
- The dual graph must also be also simple
First, rewrite the CS theory in the continuum

\[
S = \frac{k}{4\pi} \int d\vec{r} dt \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda = \frac{k}{2\pi} \int d\vec{r} dt \left[ A_0 \nabla \times \vec{A} - \frac{1}{2} \vec{A} \times \frac{d\vec{A}}{dt} \right]
\]

\[
S = \frac{k}{2\pi} \int d\vec{r} dt \left[ A_0 B - \frac{1}{2} \epsilon^{ij} A_i \dot{A}_j \right]
\]

The discrete version (we sum over repeated indices):

\[
S = \frac{k}{2\pi} \int dt \left[ A_0 M_{vf} \Phi_f - \frac{1}{2} K_{e,er} A_e \dot{A}_e \right]
\]

We need to define two matrices M and K to contract all subindices
Flux Attachment

$$S = \frac{k}{2 \pi} \int d\vec{r} dt \left[ A_0 B - \frac{1}{2} \epsilon^{ij} A_i \dot{A}_j \right]$$

- The first term: flux attachment: any charged particle carries some magnetic flux quanta.
  
  Proof:
  $$\rho(\vec{r}, t) = \frac{\delta S}{\delta A_0(\vec{r}, t)} = \frac{k}{2 \pi} B(\vec{r}, t)$$
  Integrate in real space in both sides
  $$q = k \frac{\Phi_B}{2 \pi}$$
  - Continuum: $q$ and $\Phi_B$ has the same location (flux attachment)
  - Discrete: $q$ on vertices but $\Phi_B$ on faces. **Need certain rule to connect $v$s with $f$s.**
    - the flux attachment rule is *local*
      - For a charged particle at $v$, the corresponding flux only occupy one face
      - The vertex and the face need to be adjacent
    - Two vertices cannot be associated with the same face
      - The flux attachment rule is a injective (one-to-one) mapping from vertices to faces
      - An *Injective mapping* implies $N_v \leq N_f$
  - The mapping is coded in the $M$ matrix
  $$S = \frac{k}{2 \pi} \int dt \left[ A_v M_{vf} \Phi_f - \frac{1}{2} K_{e,e'} A_e \dot{A}_{e'} \right]$$
  where
  $$M_{vf} = \begin{cases} 1 & \text{if } f \text{ is the image of } v \\ 0 & \text{otherwise} \end{cases}$$
The second term: the dynamics of the vector potentials (i.e. the commutation relation)

Consider $A_i$ as the canonical coordinate, the corresponding canonical momentum is

$$\pi(\vec{r}, t) = \frac{\delta S}{\delta \dot{A}_i(\vec{r}, t)} = \frac{k}{2\pi} \varepsilon^{ij} A_j(\vec{r}, t)$$

Canonical commutation relation: $[A_i(\vec{r}), A_j(\vec{r}')] = \frac{2\pi i}{k} \varepsilon^{ij} \delta(\vec{r} - \vec{r}')$

In a discrete system, $S = \frac{k}{2\pi} \int dt \left[ A_v M_{vf} \Phi_f - \frac{1}{2} K_{e,e'} A_e \dot{A}_{e'} \right]$, we have

$$[A_e, A_{e'}] = -\frac{2\pi i}{k} K^{-1} e, e'$$

Key ingredients:

- the flux attachment rule is local (need to choose a good $M$ matrix)
- Gauge invariance (neither of the two terms in the action is gauge invariant!)
- The $K$ matrix is invertible (no singularity in the commutator)
- The $K$ matrix must be local (the action is local)
- Inverse of $K$ must also be local (the commutators are local)
- Can work for lattices and graphs (no translational symmetry)
- Can work on a torus as well as any generic 2D compact manifold
- Recover the topological commutation relation between Wilson loops
Line integral of the vector potential

Consider two paths (lines) in the real space $P$ and $P'$:

$$\left[ \int_P \mathbf{d}\mathbf{r} \cdot \mathbf{A}, \int_{P'} \mathbf{d}\mathbf{r} \cdot \mathbf{A} \right] = \frac{2\pi i}{k} \nu[P, P']$$

where $\nu[P, P']$ is the oriented crossing number between the two paths (+/-1 for a crossing depending on its orientation)

Two direct implications:

- If $P$ and $P'$ are both contractible loops, the commutator must be zero.
  - Magnetic fluxes always commute with each other
  - This is because fluxes are associated with charges. Charges on different sites must commute with each other, so does the fluxes.

- If $P$ and $P'$ are two non-contractible loops (e.g. large circles for a torus), the commutator can be non-zero
  - This is the reason why a 1/3 fractional quantum Hall state have three degenerate ground states on a torus instead of $3^2=9$.

Must be preserved in the discretized theory!
The sufficient condition

Sufficient condition for the construction of a discrete CS gauge theory: There exists a one-to-one correspondence between faces and vertices such that each face is adjacent to its corresponding vertex.

- When we discuss flux attachment, we have assumed that
  - A local mapping from vertices to faces ($v$ and its image $f$ are adjacent to each other)
  - Injective ($N_v \leq N_f$)
- Here is a stronger condition: one-to-one correspondence
  - $N_v = N_f$
  - injective + surjective
Q: if we have a graph (lattice), how do we know whether the local one-to-one correspondence can be defined or not?
A: a necessary and sufficient criterion:
- For the whole system: $N_v = N_f$
- For any subgraph: $N_{v'} \geq N_{f'}$

Hall’s marriage theorem
- Consider a group of girls and boys.
- Connect a boy and a girl, if they know each other.
- Now, try to marriage each girl with one boy that she knows (and knows her)
- One girl (boy) can only marry one boy (girl)
What is the necessary and sufficient condition to ensure that all girls can get married?
    For each subset of $k$ girls, these girls know at least $k$ boys

We can map faces to girls and vertices to boys:
- Each subgraph must have $N_{v'} \geq N_{f'}$, to make sure that all faces can be “married” to neighboring vertices.
- Because $N_v = N_f$, when all vertices are married, it implies that all faces also get married (a one-to-one correspondence).
Some lattice examples

(a) $N_v < N_f$
(b) $N_v > N_f$
(c) $N_f = N_v$

Works

(d) Doesn’t work
Construct the action

\[ S = \frac{k}{2\pi} \int dt \left[ A_v M_{vf} \Phi_f - \frac{1}{2} K_{e,er} A_e \dot{A}_e \right] \]

- Once we have a one-to-one local correspondence between \( v \) and \( f \), the \( M \) matrix is determined immediately.

- The \( K \) matrix is

\[ K_{e,er} = \begin{cases} 
\pm 1 & \text{if } e \text{ and } e' \text{ belongs to the same face} \\
0 & \text{otherwise}
\end{cases} \]

The sign is determined as this:
- Mark the vertex \( v \) that is attached with this face (the circle)
- Go around the face counterclockwise from \( e \) to \( e' \),
- \(+1\), if we pass through the special vertex \( v \),
- \(-1\), if we don’t
- Here we assume \( e \) and \( e' \) have the same orientation.
- If not, we will get another factor of \(-1\)

- \( K \) is local! nonzero only if \( e \) and \( e' \) belongs to the same face
The dual graph

It is easy to realize that if a local one-to-one correspondence exist for a graph, so does the dual graph.

So we can repeat the same construction for the dual graph to create a Chern-Simons theory on the dual graph, using the same $M$ matrix (the same flux attachment rule). There we get a dual matrix $K^\ast$.

We find that $KK^\ast = K^\ast K = -1$, which has two implications:

- $K$ is invertible
- $K^{-1} = -K^\ast$

It is very important make sure that $K$ is invertible (nonsingular), because

$$[A_e, A_{e^\prime}] = -\frac{2\pi i}{k} K^{-1}_{e,e^\prime}$$

- $K^{-1}$ is local (only nonzero when $e$ and $e^\prime$ share the same ending point)
- Thus, the commutator is local
Commutation relation for line integrals

In the continuum, we know that:

$$\left[ \int_P d\vec{r} \cdot \vec{A}, \int_{P'} d\vec{r} \cdot \vec{A} \right] = \frac{2\pi i}{k} \nu[P, P']$$

In our discrete theory, we can represent a path \((P)\) on the graph using a \(N_e\)-dimensional vector:

$$\xi_P = (0,0,1,...,-1,0,0,...,0)$$

- \(N_e\) the number of edges in the graph
- Each component represents an edge of the graph
- If the path \(P\) contains the edge \(e\), and the direction of \(e\) is along the path, the \(e\)th component \(\xi_{P,e} = +1\)
- If the path \(P\) contains the edge \(e\), and the direction of \(e\) is opposite to the path, the \(e\)th component \(\xi_{P,e} = -1\)
- Otherwise, \(\xi_{P,e} = 0\)

Because \([A_e, A_{e'}] = -\frac{2\pi i}{k} K^{-1}_{e,e'}\), the commutator between two line integrals is

$$\left[ \xi_{P,e} A_e, \xi_{P',e'} A_{e'} \right] = -\frac{2\pi i}{k} K^{-1}_{e,e'} \xi_{P,e} \xi_{P',e'}$$

One can prove that our \(K\) matrix satisfies

$$-\frac{2\pi i}{k} K^{-1}_{e,e'} \xi_{P,e} \xi_{P',e'} = \frac{2\pi i}{k} \nu[P, P']$$

The topological commutation relation is recovered
Why $N_v = N_f$

- When we discuss the flux attachment, we have assumed that
  - A local mapping from vertices to faces ($v$ and its image $f$ are adjacent to each other)
  - Injective ($N_v \leq N_f$)
- Why $N_v = N_f$?
  - This is because the $K$ matrix must be nonsingular

Now, we prove that the $K$ matrix is singular for $N_v < N_f$, and thus $N_v = N_f$. 
A brief introduction to the algebra graph theory

- **The edge space**
  - We can use a $N_e$-dimensional vector to represent each edge: $(0,0,0, ..., 1, ..., 0,0)$
  - All these vectors span an $N_e$-dimensional linear space, known as the edge space
  - The $K$ matrix is a rank-2 tensor defined in this line space

- **Circuit-subspace: all possible loops**
  - Each (contractible/non-contractible) loop can be represented by a vector
    \[ \xi_P = (0,0, +1, ..., -1,0,0, ... 0) \]
  - All linear independent loops span a subspace of the edge space, known as the circuit space

- **Cut-subspace: all possible way to cut the graph**
  - Separate all vertices into two subsets (A and B)
  - All edges between A and B can be represented by a vector
    \[ \xi_C = (0,0, +1, ..., -1,0,0, ... 0) \]
  - For all possible ways to separate vertices into two subsets, these vectors form a subspace of the edges, known as the cut-space

- These two subspaces are orthogonal
- The edge space is the direct sum of the two subspaces
- A rank-2 tensor on the edge space can be written into $2\times2$ blocks
  \[
  \begin{pmatrix}
  X & Y \\
  Z & W
  \end{pmatrix}
  \]
Manifold with zero genus ($g = 0$)

- Circuit-subspace (dimension $N_f-1$):
  A complete set of independent contractible loops

- Cut-subspace (dimension $N_v-1$):
  A complete set of independent contractible loops in the dual graph

Now I choose my basis such that the first $N_f-1$ vectors come from the circuit subspace, and the rest come from the cut-subspace:

- A rank-2 tensor in the edge space (e.g. the $K^{-1}$ matrix) takes a block form
  
  $$K^{-1} = \begin{pmatrix} 0 & A \\ -A^T & B \end{pmatrix}$$

  - The upper-left corner: a $(N_f - 1) \times (N_f - 1)$-matrix
  - The lower-right corner: a $(N_v - 1) \times (N_v - 1)$ matrix
  - For the $K^{-1}$-matrix, the upper-left block is zero, because
    
    $$[\xi_{P,e}A_{e}, \xi_{P',e'}A_{e'}] = -\frac{2\pi i}{k} K^{-1}_{e,e'}\xi_{P,e}\xi_{P',e'} = 0$$
    
    for any two contractible loops.

- If the zero block is larger than the lower-right block $B$, $\det K^{-1} = 0$

- To avoid singularity, $N_v$ can NOT be smaller than $N_f$
Manifold with nonzero genus \((g > 0)\)

- **Subspace I** (dimension \(N_f - 1 + g\)):
  - A complete set of independent contractible loops
  - \(g\) non-contractible loops, which don’t cross with each other

- **Subspace II** (dimension \(N_v - 1 + g\)):
  - A complete set of independent contractible loops in the dual graph
  - The other \(g\) non-contractible loops

Now I choose my basis such that the first \(N_f - 1 + g\) vectors come from the subspace I, and the rest come from the subspace II:

- The \(K^{-1}\) matrix takes a block form
  \[
  K^{-1} = \begin{pmatrix}
  0 & A \\
  -A^T & B
  \end{pmatrix}
  \]
  - The upper-left corner: a \((N_f - 1 + g) \times (N_f - 1 + g)\)-matrix
  - The lower-right corner: a \((N_v - 1 + g) \times (N_v - 1 + g)\)-matrix
  - The upper-left block is zero, because of the commutation relation

- If the zero block is larger than the lower-right block \(B\), \(\det K^{-1} = 0\)
- To avoid singularity, \(N_v\) can NOT be smaller than \(N_f\)
\[ S = \frac{k}{2\pi} \int dt \left[ A_v M_{vf} \Phi_f - \frac{1}{2} K_{e,e} A_e \dot{A}_e \right] \]

- The flux attachment (the M matrix) requires \( N_v \leq N_f \)
- The dynamics (the K matrix) requires \( N_v \geq N_f \)
- \( N_v = N_f \)

- To have a discrete CS theory shown above, a necessary and sufficient condition is:

  *There exists a one-to-one correspondence between faces and vertices such that each face is adjacent to its corresponding vertex.*

- Equivalently:
  - For the whole system: \( N_v = N_f \)
  - For any subgraph: \( N_{v'} \geq N_{f'} \)
Consider a 2D planar lattice embedded on an arbitrary compact 2D manifold

The charge conservation law: \( \partial_t \rho_v + D_{v,e} j_e = 0 \)

where \( D_{v,e} \) is the incidence matrix (spatial derivative on a graph)

\[
D_{v,e} = \begin{cases} 
+1 & \text{if } v \text{ is the positive endpoint of } e \\
-1 & \text{if } v \text{ is the negative endpoint of } e \\
0 & \text{if } v \text{ is not an endpoint of } e
\end{cases}
\]

Because of the charge conservation law, we can define a gauge field on the dual lattice \( a^* \)

\( \rho_v = \frac{1}{2\pi} \xi_{f^*,e^*} a_{e^*}^*, \) where \( \xi_{f^*,e^*} = \pm 1 \) if \( e^* \) is an edge of the face \( f^* \).

\( j_e = \frac{1}{2\pi} (D_{v^*,e^*} a^*_{v^*} - \partial_t a^*_{e^*}) \), where \( D_{v^*,e^*} \) is the incidence matrix of the dual graph

They are discrete version of \( \rho = \frac{1}{2\pi} \varepsilon_{i,j} \partial_i a_j \) and \( j_i = \frac{1}{2\pi} \varepsilon_{i,j} (\partial_j a_0 - \partial_t a_j) \)

The current and density are invariant under a gauge transformation for \( a^* \)

If single-particle excitations are gapped out (i.e. an insulator), the low energy physics is described by \( \rho_v \) and \( j_e \), and thus described by a gauge theory of \( a^* \)

If the action for \( a^* \) is a CS gauge theory, the system is a FQH state with Hall conductivity \( 1/k^* \)
Hydra-dynamic theory of the FCI

- Couples $a^*$ to the electro-magnetic gauge field $A$

  $$ S_{CS} = \frac{k^*}{2\pi} \int dt \left( a^*_v M^*_v, f^* \xi^*_f, e^* a^* e^* - \frac{1}{2} a^*_e K^*_e, e^* \dot{a}^*_e e^* \right) $$

  $$ S_{coupling} = \int dt (\rho_v A_v - j_e A_e) = \int \frac{dt}{2\pi} \left( \xi^*_f, e^* a^*_e A_v \delta^*_f, v - D^*_v, e^* a^*_v A_e \delta_e, e^* + \partial_0 a^*_e A_e \delta_e, e^* \right) $$

- Integrate out $a^*$, and obtain an effective theory for $A$
  - Can be done exactly, because the theory is quadratic
  - Can be done analytically, because we know the inverse matrix of $K$

- Final result: a CS gauge theory for $A$ defined on the original graph, with coefficient $1/k^*$

  $$ S = \frac{1/k^*}{2\pi} \int dt \left( A_v M_v, f \Phi_f - \frac{1}{2} A_e K_e, e^* \dot{A}_e e^* \right) $$
What is the smallest system to support the FQHE?

A Tetrahedron:
- A 2D planar graph embedded on a sphere
- Same number of vertices and faces
- Can define a local one-to-one correspondence between vertices and faces
- Self-dual

The M matrix:

\[
M = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

The K Matrix:

\[
K = \frac{1}{2} \begin{pmatrix}
0 & -1 & -1 & 0 & +1 & +1 \\
+1 & 0 & +1 & -1 & 0 & +1 \\
+1 & -1 & 0 & +1 & -1 & 0 \\
0 & +1 & -1 & 0 & -1 & +1 \\
-1 & 0 & +1 & +1 & 0 & +1 \\
-1 & -1 & 0 & -1 & -1 & 0
\end{pmatrix}
\]

\[
S = \frac{k}{2 \pi} \int dt \left[ A_v M_{vf} \Phi_f - \frac{1}{2} K_{e,e'} A_e \dot{A}_{e'} \right]
\]
We find

- A discretized Chern-Simons gauge theory on arbitrary planar lattices embedded on an arbitrary 2D compact manifold
- As long as $N_v = N_f$ for the graph and $N_{v'} \geq N_f$, for every subgraph, we can construct a local one-to-one correspondence between faces and vertices (i.e. a local flux attachment rule)
- Then, we can write done a discretized Chern-Simons gauge theory
  - Quadratic
  - Local (the theory as well as the commutator)
  - Gauge invariance
  - Non-singular (the K-matrix is invertible)
  - Preserve the topological commutation relation
  - Dual lattice has the dual theory
- The condition above is not only sufficient but also necessary

Applications:

- Topological field theory for fractional Chern insulators
- Tetrahedron: smallest setup for a CS gauge theory
- Chiral spin liquid: (Kumar, Sun and Fradkin, arXiv:1409.2171)