A discretized Chern-Simons gauge theory on arbitrary graphs

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Outline

- How to construct a discretized Chern-Simons gauge theory
  - A necessary and sufficient condition for such a theory to be simple (quadratic, local, etc.)
- Verify that the discretized theory preserves all necessary properties of the CS theory in the continuum
- Application I: the topological field theory for fractional Chern insulators
  - What is the smallest setup to support a FQH system
- Application II: Chiral spin liquids

The Chern-Simons gauge theory

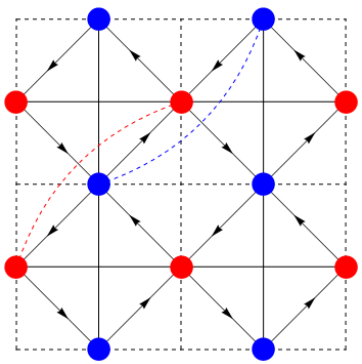
$$S = \frac{k}{4\,\pi} \int d\vec{r} dt\, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

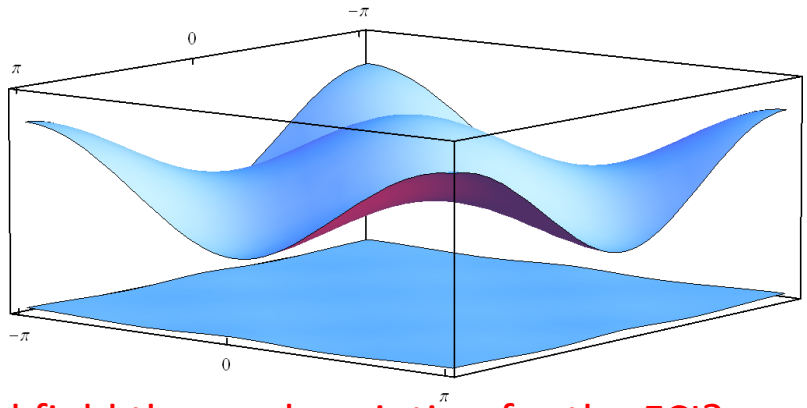
- A topological field theory
  - Quantized coefficient: k is an integer
  - Topological degeneracy on manifold with non-zero genus
- Integer and fractional quantum Hall effect
  - The response theory for the integer quantum Hall effect
  - The topological field theory for the fractional quantum Hall effect
- Chiral spin liquids

- A quadratic theory
- Written in terms of A, instead of E and B (be careful about gauge invariance)

Why discretizing?

- Fractional Chern insulators (FQHE in lattice systems with B=0)





Tang, et. al., PRL (2011).
KS, et. al., PRL (2011).
Neupert, et. al., PRL (2011).
Sheng, Nat Comm, (2011)

Q: What is the topological field theory description for the FCI?

- Chiral Spin liquid in frustrated spin systems
  - Spin-1/2 doesn't follow the canonical commutation relation
  - Can be mapped into hard-core bosons (with infinite interactions)
  - Map into fermions?
    - Don't need infinite interactions
    - But spins are bosons, not fermions
    - 1D: Fermionization (the Jordan-Wigner transformation)
    - 2D: Fermions coupled with a C-S gauge field (Fradkin 1989 and 1994)

For frustrated spin systems, the lattice structure plays a crucial role.

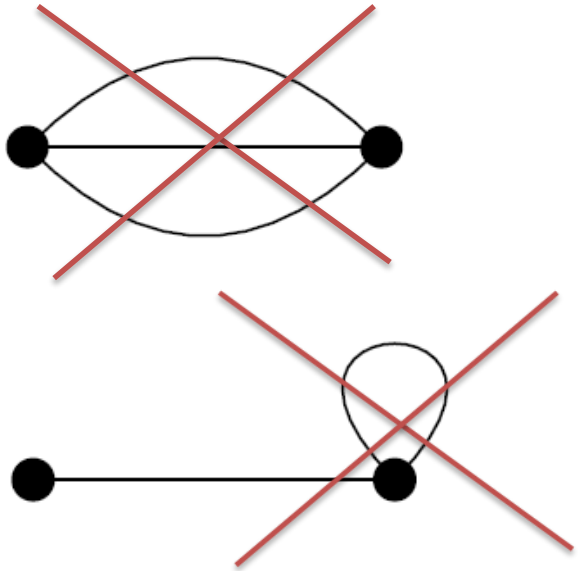
A CS theory on a simple square lattice embedded on a torus (PBCs).
Fradkin 1989. Eliezer and Semenoff 1992. Lopez, Rojo, Fradkin 1994

Objectives

- Discretizing the space (not time)
- A quadratic theory (same as the continuum)
- Doesn't require translational symmetries: lattices and graphs (important for the QHE)
- Can be embedded on arbitrary 2D manifolds with arbitrary genus (important for the QHE)
- Shall preserve all topological properties of the CS theory in the continuum

Constrains on the lattices (graphs):

- Must be planar (no crossing bonds)
- Must be simple
  - Only 1 bond between any two neighboring sites
  - No loop with length 1
- The dual graph must also be also simple



Gauge theory on a lattice

Sites (vertices)	Bonds (edges)	Plaquettes (faces)
$A_0\ (A_v)$	$\vec{A}\ \ (A_e)$	$\Phi\ (\Phi_f)$
ρ	$\vec{j}$	

- First, rewrite the CS theory in the continuum

$$S = \frac{k}{4\,\pi}\int d\vec{r}dt\ \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda = \frac{k}{2\,\pi}\int d\vec{r}dt\ [A_0\nabla\times\vec{A} - \frac{1}{2}\vec{A}\times\frac{d\vec{A}}{dt}]$$

$$S = \frac{k}{2\,\pi}\int d\vec{r}dt\ [A_0 B - \frac{1}{2}\epsilon^{ij}A_i\dot{A}_j]$$

- The discrete version (we sum over repeated indices):

$$S = \frac{k}{2\,\pi}\int dt\ [A_v M_{vf}\Phi_f\ \ - \frac{1}{2}K_{e,e'}A_e\dot{A}_{e'}]$$

We need to define two matrices M and K to contract all subindices

Flux Attachment

$$S = \frac{k}{2\pi}\int d\vec{r}dt\,[A_0 B - \frac{1}{2}\epsilon^{ij}A_i\dot{A}_j]$$

- The first term: flux attachment: any charged particle carriers some magnetic flux quanta.

  Proof: $$\rho(\vec{r},t) = \frac{\delta S}{\delta A_0(\vec{r},t)} = \frac{k}{2\pi}B(\vec{r},t)$$

  Integrate in real space in both sides

  $$q = k\frac{\Phi_B}{2\pi}$$

  - Continuum: q and Φ_B has the same location (flux attachment)
- Discrete: q on vertices but Φ_B on faces. Need certain rule to connect vs with fs.
  - the flux attachment rule is *local*
    - For a charged particle at v, the corresponding flux only occupy one face
    - The vertex and the face need to be adjacent
  - Two vertices cannot be associated with the same face
    - The flux attachment rule is a injective (one-to-one) mapping from vertices to faces
    - An *Injective mapping* implies $N_v \leq N_f$
- The mapping is coded in the M matrix

$$S = \frac{k}{2\pi}\int dt\,[A_v M_{vf}\Phi_f - \frac{1}{2}K_{e,e'}A_e\dot{A}_{e'}] \qquad \text{where} \qquad M_{vf} = \begin{cases} 1 & \text{if } f \text{ is the image of } v \\ 0 & \text{otherwise} \end{cases}$$

Dynamics

$$S = \frac{k}{2\pi}\int d\vec{r}dt\,[A_0 B - \frac{1}{2}\epsilon^{ij}A_i\dot{A}_j]$$

- The second term: the dynamics of the vector potentials (i.e. the commutation relation)
  - Consider A_i as the canonical coordinate, the corresponding canonical momentum is

$$\pi(\vec{r},t) = \frac{\delta S}{\delta \dot{A}_i(\vec{r},t)} = \frac{k}{2\pi}\epsilon^{ij}A_j(\vec{r},t)$$

  - Canonical commutation relation: $[A_i(\vec{r}), A_j(\vec{r}')] = \frac{2\pi i}{k}\epsilon^{ij}\delta(\vec{r}-\vec{r}')$
- In a discrete system, $S = \frac{k}{2\pi}\int dt\,[A_v M_{vf}\Phi_f - \frac{1}{2}K_{e,e'}A_e\dot{A}_{e'}]$, we have

$$[A_e, A_{e'}] = -\frac{2\pi i}{k}K^{-1}{}_{e,e'}$$

- Key ingredients:
  - the flux attachment rule is *local* (need to choose a good M matrix)
  - Gauge invariance (neither of the two terms in the action is gauge invariant!)
  - The K matrix is invertible (no singularity in the commutator)
  - The K matrix must be local (the action is local)
  - Inverse of K must also be local (the commutators are local)
  - Can work for lattices and graphs (no translational symmetry)
  - Can work on a torus as well as any generic 2D compact manifold
  - Recover the topological commutation relation between Wilson loops

Line integral of the vector potential

Consider two paths (lines) in the real space P and P' :

$$\left[\int_P d\vec{r}\cdot\vec{A}\,,\int_{P'} d\vec{r}\cdot\vec{A}\right]=\frac{2\pi i}{k}\nu[P,P']$$

where $\nu[P,P']$ is the oriented crossing number between the two paths (+/-1 for a crossing depending on its orientation)

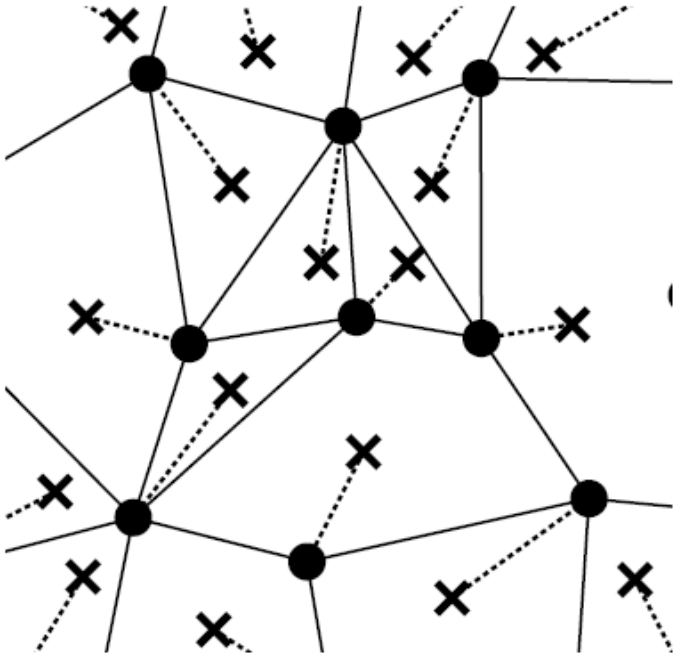
Two direct implications:

- If P and P' are both contractible loops, the commutator must be zero.
  - magnetic fluxes always commute with each other
  - This is because fluxes are associated with charges. Charges on different sites must commute with each other, so does the fluxes.
- If P and P' are two non-contractible loops (e.g. large circles for a torus), the commutator can be non-zero
  - This is the reason why a 1/3 fractional quantum Hall state have three degenerate ground states on a torus instead of 3^2=9.

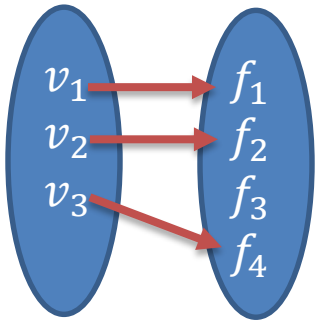
Must be preserved in the discretized theory!

The sufficient condition

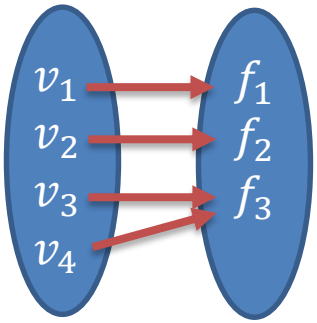
Sufficient condition for the construction of a discrete CS gauge theory: *There exists a one-to-one correspondence between faces and vertices such that each face is adjacent to its corresponding vertex.*



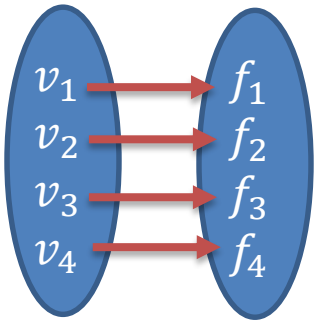
- When we discuss flux attachment, we have assumed that
  - A local mapping from vertices to faces (v and its image f are adjacent to each other)
  - Injective ($N_v \leq N_f$)
- Here is a stronger condition: *one-to-one correspondence*
  - $N_v = N_f$
  - injective + surjective



injective



surjective



one-to-one corresponds

Hall's marriage theorem

Q: if we have a graph (lattice), how do we know whether the local one-to-one correspondence can be defined or not?
A: a necessary and sufficient criterion:

- For the whole system: $N_v = N_f$
- For any subgraph: $N_{v'} \geq N_{f'}$

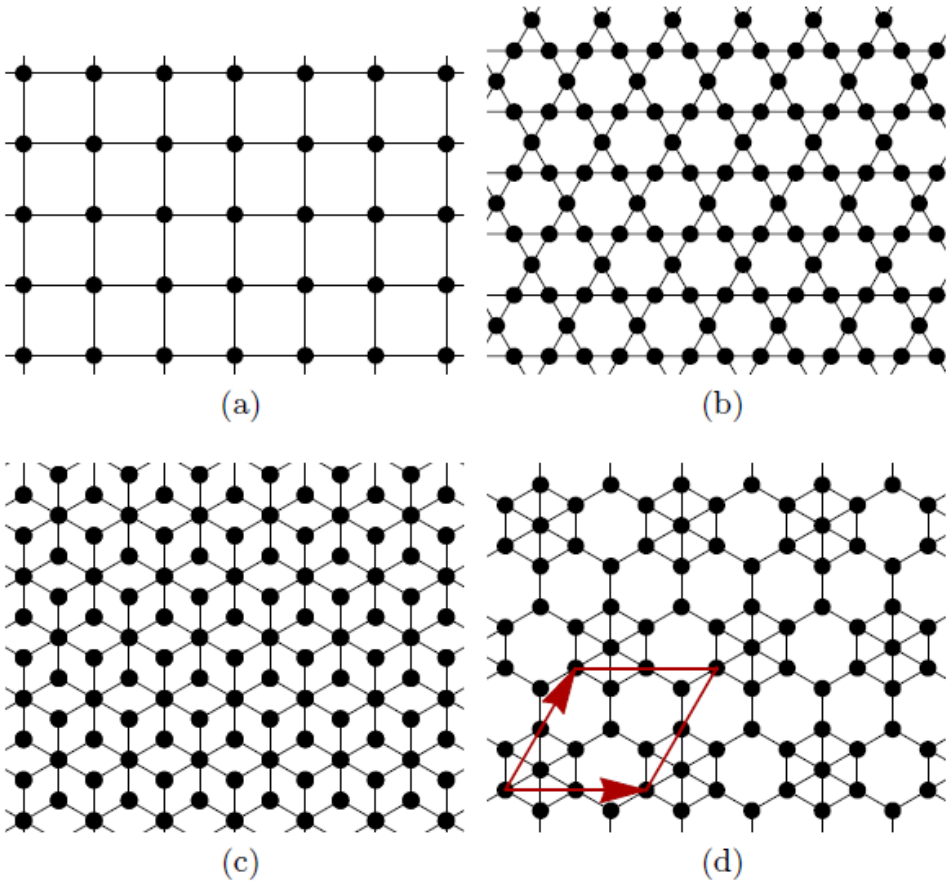
➢ Hall's marriage theorem
- Consider a group of girls and boys.
- Connect a boy and a girl, if they know each other.
- Now, try to marriage each girl with one boy that she knows (and knows her)
- One girl (boy) can only marry one boy (girl)

What is the necessary and sufficient condition to ensure that all girls can get married?

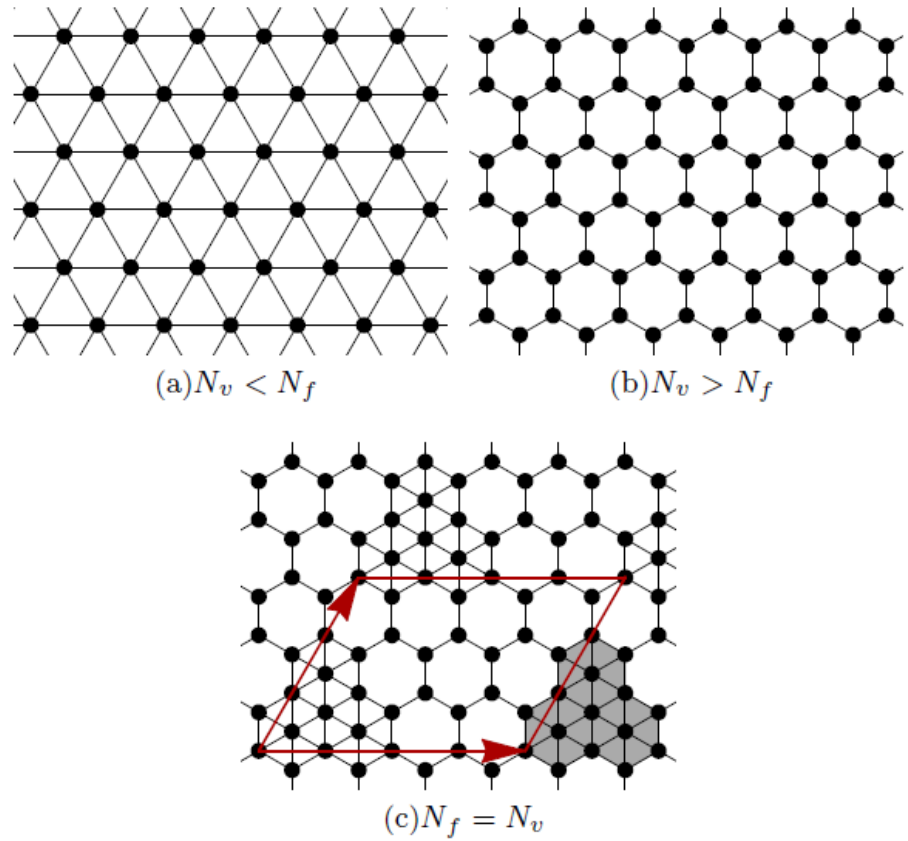
For each subset of k girls, these girls know at least k boys

➢ We can map faces to girls and vertices to boys:
- Each subgraph must have $N_{v'} \geq N_{f'}$ to make sure that all faces can be "married" to neighboring vertices.
- Because $N_v = N_f$, when all vertices are married, it implies that all faces also get married (a one-to-one correspondence).

Some lattice examples



Works



Doesn't work

Construct the action

$$S = \frac{k}{2\pi}\int dt\,[A_v M_{vf}\Phi_f - \frac{1}{2}K_{e,e'}A_e\dot{A}_{e'}]$$

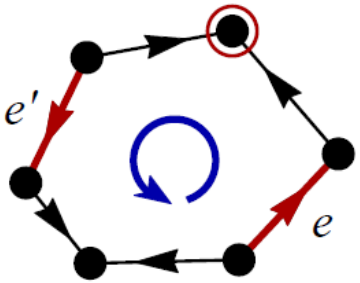
- Once we have a one-to-one local correspondence between v and f, the M matrix is determined immediately.

- The K matrix is

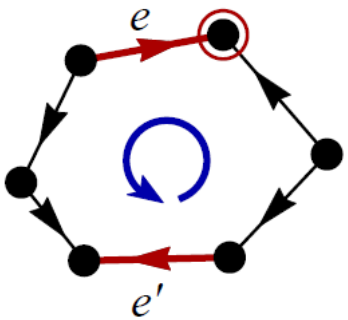
$$K_{e,e'} = \begin{cases} \pm 1 & \text{if } e \text{ and } e' \text{belongs to the same face} \\ 0 & \text{otherwise} \end{cases}$$

The sign is determined as this:

- Mark the vertex v that is attached with this face (the circle)
- Go around the face counterclockwise from e to e',
- +1, if we pass through the special vertex v,
- -1, if we don't
- Here we assume e and e' have the same orientation.
- If not, we will get another factor of -1



$K_{e,e'}$=+1



$K_{e,e'}$=−1

- K is local! nonzero only if e and e' belongs to the same face

The dual graph

It is easy to realize that if a local one-to-one correspondence exist for a graph, so does the dual graph

So we can repeat the same construction for the dual graph to create a Chern-Simons theory on the dual graph, using the same M matrix (the same flux attachment rule). There we get a dual matrix K^*.

We find that $KK^* = K^*K = -1$, which has two implications:

- K is invertible
- $K^{-1} = -K^*$

It is very important make sure that K is invertible (nonsingular), because

$$[A_e, A_{e'}] = -\frac{2\pi i}{k} K^{-1}{}_{e,e'}$$

- K^{-1} is local (only nonzero when e and e' share the same ending point)
- Thus, the commutator is local

Commutation relation for line integrals

- In the continuum, we know that:

$$\left[\int_P d\vec{r}\cdot\vec{A}\,,\int_{P'} d\vec{r}\cdot\vec{A}\right]=\frac{2\pi i}{k}\nu[P,P']$$

- In our discrete theory, we can represent a path (P) on the graph using a N_e-dimensional vector:

$$\xi_P=(0,0,1,\dots,-1,0,0,\dots 0)$$

  - N_e the number of edges in the graph
  - Each component represents an edge of the graph
  - If the path P contains the edge e, and the direction of e is along the path, the eth component $\xi_{P,e}=+1$
  - If the path P contains the edge e, and the direction of e is opposite to the path, the eth component $\xi_{P,e}=-1$
  - Otherwise, $\xi_{P,e}=0$
- Because $[A_e,A_{e'}]=-\frac{2\,\pi\,i}{k}K^{-1}{}_{e,e'}$, the commutator between two line integrals is

$$[\xi_{P,e}A_e,\xi_{P',e'}A_{e'}]=-\frac{2\,\pi\,i}{k}\,K^{-1}{}_{e,e'}\xi_{P,e}\xi_{P',e'}$$

One can prove that our K matrix satisfies $-\frac{2\,\pi\,i}{k}\,K^{-1}{}_{e,e'}\xi_{P,e}\xi_{P',e'}=\frac{2\pi i}{k}\nu[P,P']$

The topological commutation relation is recovered

Why $N_v = N_f$

- When we discuss the flux attachment, we have assumed that
  - A local mapping from vertices to faces (v and its image f are adjacent to each other)
  - Injective ($N_v \leq N_f$)
- Why $N_v = N_f$?
  - This is because the K matrix must be nonsingular

Now, we prove that the K matrix is singular for $N_v < N_f$, and thus $N_v = N_f$.

A brief introduction to the algebra graph theory

- The edge space
  - We can use a N_e-dimensional vector to represent each edge: $(0,0,0,\dots,1,\dots 0,0)$
  - All these vectors spans an N_e-dimensional linear space, known as the edge space
  - The K matrix is a rank-2 tensor defined in this line space
- Circuit-subspace: all possible loops
  - Each (contractible/non-contractible) loop can be represented by a vector

$$\xi_P = (0,0,+1,\dots,-1,0,0,\dots 0)$$

  - All linear independent loops span a subspace of the edge space, known as the circuit space
- Cut-subspace: all possible way to cut the graph
  - Separate all vertices into two subsets (A and B)
  - All edges between A and B can be represented by a vector

$$\xi_C = (0,0,+1,\dots,-1,0,0,\dots 0)$$

  - For all possible ways to separate vertices into two subsets, these vectors form a subspace of the edges, known as the cut-space

- These two subspaces are orthogonal
- The edge space is the direct sum of the two subspaces
- A rank-2 tensor on the edge space can be written into 2X2 blocks

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

Manifold with zero genus ($g = 0$)

- Circuit-subspace (dimension N_f-1):
  A complete set of independent contractible loops
- Cut-subspace (dimension N_v-1):
  A complete set of independent contractible loops in the dual graph

Now I choose my basis such that the first N_f-1 vectors come from the circuit subspace, and the rest come from the cut-subspace:

- A rank-2 tensor in the edge space (e.g. the K^{-1} matrix) takes a block form

$$K^{-1} = \begin{pmatrix} 0 & A \\ -A^T & B \end{pmatrix}$$

  - The upper-left corner: a $(N_f - 1) \times (N_f - 1)$-matrix
  - The lower-right corner: a $(N_v - 1) \times (N_v - 1)$matrix
  - For the K^{-1}-matrix, the upper-left block is zero, because

$$[\xi_{P,e} A_e, \xi_{P',e'} A_{e'}] = -\frac{2\pi i}{k} K^{-1}{}_{e,e'} \xi_{P,e} \xi_{P',e'} = 0$$

    for any two contractible loops.
- If the zero block is larger than the lower-right block B, $\det K^{-1} = 0$
- To avoid singularity, N_v can NOT be smaller than N_f

Manifold with nonzero genus ($g > 0$)

- Subspace I (dimension $N_f - 1 + g$):
  - A complete set of independent contractible loops
  - g non-contractible loops, which don't cross with each other

- Subspace II (dimension $N_v - 1 + g$):
  - A complete set of independent contractible loops in the dual graph
  - The other g non-contractible loops

Now I choose my basis such that the first $N_f - 1 + g$ vectors come from the subspace I, and the rest come from the subspace II:

- The K^{-1} matrix takes a block form

$$K^{-1} = \begin{pmatrix} 0 & A \\ -A^T & B \end{pmatrix}$$

  - The upper-left corner: a $(N_f - 1 + g) \times (N_f - 1 + g)$-matrix
  - The lower-right corner: a $(N_v - 1 + g) \times (N_v - 1 + g)$-matrix
  - The upper-left block is zero, because of the commutation relation

- If the zero block is larger than the lower-right block B, $\det K^{-1} = 0$
- To avoid singularity, N_v can NOT be smaller than N_f

Summary

$$S = \frac{k}{2\pi}\int dt\,[A_v M_{vf}\Phi_f \;-\frac{1}{2}K_{e,e'}A_e\dot{A}_{e'}]$$

- The flux attachment (the M matrix) requires $N_v \leq N_f$
- The dynamics (the K matrix) requires $N_v \geq N_f$
- $N_v = N_f$

- To have a discrete CS theory shown above, a necessary and sufficient condition is:
  There exists a one-to-one correspondence between faces and vertices such that each face is adjacent to its corresponding vertex.

- Equivalently:
  - For the whole system: $N_v = N_f$
  - For any subgraph: $N_{v'} \geq N_{f'}$

Hydra-dynamic theory of the FCI

Consider a 2D planar lattice embedded on an arbitrary compact 2D manifold

- The charge conservation law: $\partial_t \rho_v + D_{v,e}\, j_e = 0$

  where $D_{v,e}$ is the incidence matrix (spatial derivative on a graph)

$$D_{v,e} = \begin{cases} +1 & \text{if } v \text{ is the postive endpoint of } e \\ -1 & \text{if } v \text{ is the negative endpoint of } e \\ 0 & v \text{ is not an endpoint of } e \end{cases}$$

- Because of the charge conservation law, we can define a gauge field on the dual lattice a^*
  - $\rho_v = \frac{1}{2\pi}\xi_{f^*,e^*}\, a^*_{e^*}$, where $\xi_{f^*,e^*} = \pm 1$ if e^* is an edge of the face f^*.
  - $j_e = \frac{1}{2\pi}(D_{v^*,e^*}\, a^*_{v^*} - \partial_t a^*_{e^*})$, where D_{v^*,e^*} is the incidence matrix of the dual graph
  - They are discrete version of $\rho = \frac{1}{2\pi}\epsilon_{i,j}\, \partial_i a_jandj_i = \frac{1}{2\pi}\epsilon_{i,j}(\partial_j a_0 - \partial_t a_j)$
  - The current and density are invariant under a gauge transformation for a^*
  - If single-particle excitations are gapped out (i.e. an insulator), the low energy physics is described by ρ_v and j_e, and thus described by a gauge theory of a^*
- If the action for a^* is a CS gauge theory, the system is a FQH state with Hall conductivity $1/k^*$

$$S_{\mathrm{CS}} = \frac{k^*}{2\pi}\int dt \left(a^*_{v^*} M^*_{v^*,f^*} \xi^*_{f^*,e*} a^*_{e^*} - \frac{1}{2} a^*_{e^*} K^*_{e^*,e'^*} \dot{a}^*_{e'^*} \right)$$

Hydra-dynamic theory of the FCI

- Couples a^* to the electro-magnetic gauge field A

$$S_{\mathrm{CS}} = \frac{k^*}{2\pi} \int dt \left(a^*_{v^*} M^*_{v^*,f^*} \xi^*_{f^*,e*} a^*_{e^*} - \frac{1}{2} a^*_{e^*} K^*_{e^*,e'^*} \dot{a}^*_{e'^*} \right)$$

$$S_{\mathrm{coupling}} = \int dt \left(\rho_v A_v - j_e A_e \right) = \int \frac{dt}{2\pi} \left(\xi^*_{f^*,e^*} a^*_{e^*} A_v \delta_{f^*,v} - D^*_{v^*,e^*} a^*_{v^*} A_e \delta_{e,e^*} + \partial_0 a^*_{e^*} A_e \delta_{e,e^*} \right)$$

- Integrate out a^*, and obtain an effective theory for A
  - Can be done exactly, because the theory is quadratic
  - Can be done analytically, because we know the inverse matrix of K

- Final result: a CS gauge theory for A defined on the original graph, with coefficient $1/k^*$

$$S = \frac{1/k^*}{2\pi} \int dt \left(A_v M_{v,f} \Phi_f - \frac{1}{2} A_e K_{e,e'} \dot{A}_{e'} \right)$$

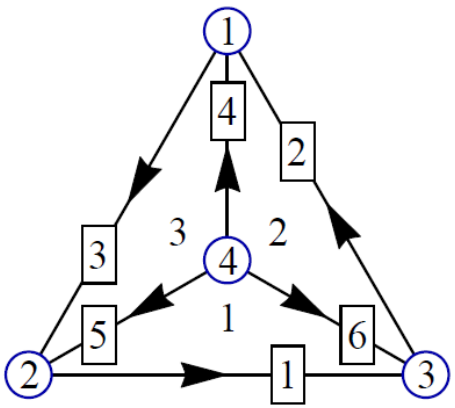
What is the smallest system to support the FQHE?

A Tetrahedron:

- A 2D planar graph embedded on a sphere
- Same number of vertices and faces
- Can define a local one-to-one correspondence between vertices and faces
- Self-dual

The M matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$



The K Matrix:

$$K = \frac{1}{2} \begin{pmatrix} 0 & -1 & -1 & 0 & +1 & +1 \\ +1 & 0 & +1 & -1 & 0 & +1 \\ +1 & -1 & 0 & +1 & -1 & 0 \\ 0 & +1 & -1 & 0 & -1 & +1 \\ -1 & 0 & +1 & +1 & 0 & +1 \\ -1 & -1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

$$S = \frac{k}{2\,\pi} \int dt\, [A_v M_{vf} \Phi_f \; - \frac{1}{2} K_{e,e'} A_e \dot{A}_{e'}]$$

Summary

We find

- A discretized Chern-Simons gauge theory on arbitrary planar lattices embedded on an arbitrary 2D compact manifold
- As long as $N_v = N_f$ for the graph and $N_{v'} \geq N_{f'}$ for every subgraph, we can construct a local one-to-one correspondence between faces and vertices (i.e. a local flux attachment rule)
- Then, we can write done a discretized Chern-Simons gauge theory
    - Quadratic
    - Local (the theory as well as the commutator)
    - Gauge invariance
    - Non-singular (the K-matrix is invertible)
    - Preserve the topological commutation relation
    - Dual lattice has the dual theory
- The condition above is not only sufficient but also necessary

Applications:

- Topological field theory for fractional Chern insulators
- Tetrahedron: smallest setup for a CS gauge theory
- Chiral spin liquid: (Kumar, Sun and Fradkin, **arXiv:1409.2171**)