Symmetry-protected topological phases: Projective construction & Bulk field theory

Péng Yè

SPT enthusiasts @ 3.14159
Collaboration with great colleagues, friends below:

![Collaborators](image)

**Focused references:**


**Related but not included here:**

- Proj. constr. of 3d SPT: 1303.3572 [see my Princeton talk in Mar.];
- Response theory: 1306.3695;
- 1d SPT order of doped spin chain: 1310.6496;
- 3d SPT: solvable lattice model (Ye, Gu, in preparation)
Preface-[1]: Triviality
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Defining triviality is quite nontrivial. → Our development of physics is to continuously redefine triviality by deeper and deeper probes / insights.
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Indeed condensing topological defects generically drives a phase transition to exotic states of matter out from a conventional state.
SPT@π
Peng Ye

Contents

1 General review

2 Projective construction of 2d SPT

3 3d SPT bulk field theory with $b \wedge b$ term

4 Summary
Section 1 General review
Hundred years’ **hard** Condensed Matter (Solid State)

Physics in progress... Great triumph in 1980s.
A central concept: “order”, a way of organization in many-body system.

1. What is the “order” that organizes the observed phenomena?
2. How to classify those “orders”?
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**Symmetry-breaking order** $G \rightarrow H$

- Symmetry-breaking order (Landau theory) works very well for a long time.
- Local order parameter, as a function, is determined by $G/H$. 
Experimental conditions: **Clean** 2DEG ($\mu \sim 10^5 \text{cm}^2 / \text{V} \cdot \text{s}$), **low** electron density ($n \sim 10^{11} / \text{cm}^2$); **low** temperature, **high** magnetic field.

Unexpected results: Wigner crystal or CDW? in Prof. Tsui’s seminal paper. Now we know: **Topological order**—(Wen) non-symmetry-breaking order; **long-range entanglement**. (Gu, Wen, Chen, et.al.) $\rightarrow$ quantum information, computing,...
Symmetry-protected topological phases (SPT)

In absence of **topological order** and **symmetry-breaking order**, we have **SPT order**.
Symmetry-protected topological phases (SPT)

In absence of topological order and symmetry-breaking order, we have SPT order.

Some star examples in more realistic electronic materials: (“fermionic SPT”)

1. Topological insulator (3d, time-reversal)
2. Quantum spin Hall insulator (2d TI, time-reversal)
3. Topological crystalline insulator. (space group, see, also, S. Ryu’s talk)
4. ......

99% can be modelled by band theory, or, weak interacting but has a well-defined free-fermion counterpart.
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99% can be modelled by band theory, or, weak interacting but has a well-defined free-fermion counterpart.

Classification of free-fermion nontrivial SPT is well-known now (Ryu, Schnyder, Furusaki, Ludwig; Kitaev, ···)

Classification of interacting-fermion nontrivial SPT is under debate (Gu, Wen, Senthil, Kapustin, Ryu, Qi....many people)!!A frontier of research!!.
In boson SPT, strong interaction and strong correlation effects are necessary!
Avoid boson condensation (boson system); avoid spin order (spin paramagnets)
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Avoid boson condensation (boson system); avoid spin order (spin paramagnets)
Some examples with concrete lattice models:

1. **Haldane phase** in spin-1 AKLT (or Heisenberg) is a bosonic SPT, a short-range entangled state. (Pollmann, et.al. Gu, Wen, et.al.)


Progress in SPT

Intensive study in North America (quite incomplete list. sorry for missing.)

- Group cohomology classification and fixed-point Lagrangian (Xie Chen, Gu, Liu, Wen)
- Classification based on Chern-Simons theory (Lu, Vishwananth)
- Classification based on non-linear $\sigma$ field theory with $\theta$-term (C. Xu’s group)
- Classification based on “cobordism” (Kapustin, et.al.)
- SPT—gauge theory duality (Levin, Gu; Wen, et.al.)
- 3d bosonic TI (Vishwanath, Senthil, Metlitski, Ye, Wen, et.al.)
- 2d projective construction (Ye, Wen; Lu, Lee, et.al.)
- Boundary anomaly (S. Ryu’s group; Wang, Wen, et.al.)
- 3d field theory (Xu, Senthil; Ye, Gu; …)
Section 2 Projective construction of 2d SPT states
Projective construction in non-Abelian QH states

At least, two useful analytic ways to construct non-Abelian QH states:

1. Correlation functions in CFT: edge states (Blok, Wen, Wu, Nayak, Wilczek, Read, Rezayi, Cappelli, et.al.), topologically invariant, two different bdr (cut through plane or through time)
Projective construction in non-Abelian QH states

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2. Projective (parton) construction: both edge and bulk field theories. (Wen, Blok, Barkeshli, et.al.)
Proj. constr. of non-Abelian QH: Wen’s recipe ’99

1. **Introduce a few partons** $\psi_a$ ($a = 1, \ldots, n$), each carries $Q_a$ charge:

   \[
   \mathcal{L}_{\text{parton}} = i\psi_a^\dagger \partial_t \psi_a + \frac{1}{2m} \psi_a^\dagger (\partial_i - iQ_a A_i)^2 \psi_a
   \]

2. **Introduce an “electron” operator**: $\Psi_e = \sum_m C_m \prod_a \psi_a^{n_a(m)}(z)$, where, $n_a^{(m)} = 0, 1$, and, $\sum_a Q_a n_a^{(m)} = e$.

3. $\psi_a \rightarrow W_{ab} \psi_b$ leaves $\Psi_e$ and electron current operators $J_e$ unchanged. Thus, $W^\dagger Q W = Q$, where, $Q = \text{diag}(Q_1, Q_2, \ldots)$. $W \in G$ (gauge group).

4. **Consider** $G_c \in G$, a connected piece of $G$, which contains identity. **Thus, define an emergent gauge field** $a_\mu \in \mathcal{L}_{G_c}$, i.e. Lie algebra of $G_c$. Thus, bulk theory becomes:

   \[
   \mathcal{L} = i\psi_a^\dagger [\delta_{ab} \partial_t - i(a_0)_{ab}] \psi_b + \frac{1}{2m} \psi_a^\dagger (\partial_i - iQ a_i - ia_i)_{ab}^2 \psi_b
   \]

5. **To ensure the bulk is gapped**, one may consider partons form IQH states with filling $\nu_a = m_a$. **Integrating** partons leads to:

   \[
   S = \frac{1}{4\pi} \text{Tr}(M(a \wedge da + \frac{2}{3} a \wedge a \wedge a)) + \frac{\text{Tr}(MQ^2)}{4\pi} A \wedge dA
   \]

   where, crossing term is eliminated by demanding: $\text{Tr}(tMQ) = 0$, with arbitrary matrix $t \in \mathcal{L}_{G_c}$. $M = \text{diag}(m_1, m_2, \ldots)$

6. **Deriving edge states** (Kac-Moody algebra) via coset theory or OPE.
To understand symmetry transformation on matter fields, we preserve partons d.o.f..

- Parton field current

\[ J^I_\mu = \frac{1}{2\pi} \epsilon^{\mu \nu \lambda} \partial_\nu a^I_\lambda \]

automatically resolve \( \partial^\mu J^I_\mu = 0 \) (\( a^I \), at this step, is a non-compact, 1-form U(1) gauge connection, defined on the real axis.)
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- Projection is done by imposing a "constraint" on parton currents (i.e. fluxes) in path-integral measure:

\[
\mathcal{F}(J^1_\mu, J^2_\mu, \ldots) = 0 \rightarrow \mathcal{F}(a^1_\mu, a^2_\mu, \ldots) = 0
\]
To understand symmetry transformation on matter fields, we preserve partons d.o.f..

- Eventually, the bulk theory is described by a new $\tilde{K}$-Chern-Simons theory. *(assuming the constraint is a linear equation)*

1. If $\det \tilde{K} = 0$, then, apply $\mathbb{GL}$ transformation to separate the zero eigenvalue.
   1. *Spontaneous Symmetry-breaking order with Goldstone modes if zero eigenvalues carry symmetry charge. Monopole is suppressed. Maxwell term sets in.*
   2. *If not, zero eigenvalues are in Polyakov confined phase (monopole condensation). Directly Remove it!!*

Proj. constr. of SPT: Ye-Wen’s program ’12
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   - **Spontaneous Symmetry-breaking order with Goldstone modes** if zero eigenvalues carry symmetry charge. Monopole is suppressed. Maxwell term sets in.
   - **If not**, zero eigenvalues are in Polyakov confined phase (monopole condensation). **Directly Remove it!!**

2. **If** $|\det(\tilde{\mathcal{K}})| = 1$, **then**, this state is a gapped state without topo. order. (might be a nontrivial SPT)

3. **If** $|\det(\tilde{\mathcal{K}})| > 1$, **then**, the state is a topologically ordered Abelian state. might be a nontrivial SET
To understand symmetry transformation on matter fields, we preserve partons d.o.f..

- Eventually, the **bulk theory** is described by a new $\tilde{\mathcal{K}}$-Chern-Simons theory. *assuming the constraint is a linear equation*

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- **Edge reconstruction** *(Sometimes, chiral edge $\rightarrow$ non-chiral edge!!)*
Proj. constr. of U(1) × U(1) SPT states

Start with a spin-1 system, $S = f^{\dagger} \Gamma f$, where $\Gamma$ are three angular momentum generators. Partons: $f = (f_{+1}, f_0, f_{-1})$.

- Fill three Chern bands on a Kagome lattice (with Chern number $C_1, C_0, C_{-1}$).

- Minimal symmetry is: $\text{U}(1) \times \text{U}(1)$ (conservation of both $\sum_i S_i^z$ and $\sum_i (S_i^z)^2$)

- Projection is done by imposing a constraint on gauge fluxes: $J^{+1}_\mu + J^0_\mu + J^{-1}_\mu = K_\mu$, where, $K_\mu = (1,0,0)$. 
Proj. constr. of $U(1) \times U(1)$ SPT states

Start with a spin-1 system, $S = f^\dagger \Gamma f$, where $\Gamma$ are three angular momentum generators. Partons: $f = (f_1, f_0, f_{-1})$. 

Summary

Proj. constr. of $U(1) \times U(1)$ SPT states

Start with a spin-1 system, $S = f^\dagger \Gamma f$, where $\Gamma$ are three angular momentum generators. Partons: $f = (f_1, f_0, f_{-1})$. 

Groundstate wavefunction: $\braket{\text{free partons}}$ (Monte Carlo testable)

Edge is reconstructed to be non-chiral, with even-quantized nontrivial spin Hall conductance: $\sigma_{Sz}^H = \frac{1}{2}\pi \times 2$. 

19 / 41
Proj. constr. of $U(1) \times U(1)$ SPT states

Start with a spin-1 system, $S = f^\dagger \Gamma f$, where $\Gamma$ are three angular momentum generators. Partons: $f = (f_+, f_0, f_-)$.

Consider $C_1 = 1, C_0 = -1, C_{-1} = 1$: The free parton theory is a “fermionic” Chern-Simons theory with chiral edge ($1 - 1 + 1 \neq 0$)

$$
\mathcal{L} = \frac{1}{4\pi} \left( \begin{array}{ccc} a_1 & a_0 & a_{-1} \\ a_\mu & a_\lambda & a_{-1} \lambda \\ a_{-1} & a_0 & a_\mu \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \partial_\nu \left( \begin{array}{c} a_1^\lambda \\ a_0^\lambda \\ a_{-1}^\lambda \end{array} \right) \epsilon^{\mu \nu \lambda} + \frac{1}{2\pi} A^S_{\mu} \partial_\nu \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) \left( \begin{array}{c} a_1^\lambda \\ a_0^\lambda \\ a_{-1}^\lambda \end{array} \right) \epsilon^{\mu \nu \lambda}
$$

[1] Groundstate wavefunction: $P_G |\text{free partons}\rangle$ (Monte Carlo testable)

[2] Edge is reconstructed to be non-chiral, with even-quantized nontrivial spin Hall conductance: $\sigma_{S^z} = \frac{1}{2\pi} \times 2$. 
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$$

After Gutzwiller projection, $da^1 + da^{-1} + da^0 = 0$, we end up with:

$$
\mathcal{L} = \frac{1}{4\pi} \begin{pmatrix} a^1_\mu & a^{-1}_\mu \\ a^1_\lambda & a^{-1}_\lambda \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \partial_\nu \begin{pmatrix} a^1_\lambda \\ a^{-1}_\lambda \end{pmatrix} \epsilon^{\mu \nu \lambda} + \frac{1}{2\pi} A^S_{\mu} \partial_\nu \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} a^1_\lambda \\ a^{-1}_\lambda \end{pmatrix} \epsilon^{\mu \nu \lambda}
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$$
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$$
One coin, two sides:
Non-zero even quantized Hall response $\longleftrightarrow$ All gapping terms on the boundary necessarily break symmetry $\longleftrightarrow$ nontrivial U(1) SPT order.
Gapping conditions

Understand more mathematically:

- In terms of bosonization, perturbation terms have the form of $\sim \cos(\ell^T \phi)$
- Is there a solution $\ell$ of equation $\ell^T K^{-1} \ell = 0$? (Haldane’s null vector condition)
  - **No.** “intrinsic” state (chiral states and some nonchiral states (i.e. $\nu = 2/3$ FQH))
  - **Yes.** Is there a solution $\ell$ further satisfying $\ell^T K^{-1} q = 0$?
    - **Yes**, trivial SPT. The perturbation term gaps edge without breaking symmetry.
    - **No**, nontrivial SPT. All gapping terms break U(1) symmetry.
      → one can show: $q^T K^{-1} q$ is always even-integer (note that diagonal entries of $K$ is even for boson systems and $|\det K| = 1$).

- **A side note:**
  
  # of symmetry-breaking perturbation $\gg$ # of gapping terms.
  For example, in-plane Zeeman term can not gap edge.

$BS_x \sim \cos(2\phi_1 + \phi_{-1}) + \cos(2\phi_{-1} + \phi_1)$. 
**Numerical test**

**Figure**: Correlation function. (1) power-law decay (2) still power-law decay after in-plane Zeeman is imposed. $B_x = 0.4$
Numerics: Gapped, and, trivial topological order

Figure: Bulk correlation function. “quadropole” $Q^x = (S^x)^2 - (S^y)^2$.

Figure: Topological entanglement entropy is zero.
Short summary of proj. constr. of SPT

What have been shown:

- Proj. Constr. provides ground state wave functions of SPT order with Abelian continuous symmetry, such as $U(1) \times U(1)$. (Ye, Wen 1212.2121; Liu, Mei, Ye, Wen 1408.1676)

What have been not shown here:

1. Proj. Constr. can also construct SPT order with non-Abelian symmetry, such as $SU(2), SO(3)$ (see Ye, Wen arXiv:1212.2121)
2. In 3d, techniques become complicated (\(\because\) monopole/dyon might or might not be condensed in 3+1d gauge theory.) (see Ye, Wen arXiv:1303.3572)
Section 3 Bulk field theory of 3d SPT states
Physicists like surface

Totem and Motto of Surface Science

Figure: Physics is like a pizza, all the good stuff is on the surface! (pizza copied from Prof. Yayu Wang’s lecture note)
**However**, bulk definition is more fundamental

- Bulk theory is a "**master equation**": e.g. CS theory of QH $\rightarrow$ chiral Luttinger liquid.
However, bulk definition is more fundamental

- Bulk theory is a "master equation": e.g. CS theory of QH $\rightarrow$ chiral Luttinger liquid.

- Two concerns on "surface-only research":
  - Many-to-one correspondence between surface and bulk is generically possible.
    e.g. QH at high Landau level ($\nu=8,12$, see Cano, et al. PRB 2014)
  - Anomaly might not be fully canceled.
Review of bosonic topological insulators (BTI)

Focus on BTI only.

Symmetry group $G = U(1) \rtimes Z_2^T$ ($Z_2^T$: time-reversal symmetry, $T^2 = 1$.)

1. Classification from group cohomology with $U(1)$ coefficient:

\[ H^4(G, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2 \] (Chen, Gu, Liu, Wen 2013)
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2. Recently, people found more: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, by surface argument (Vishwanath, Senthil, PRX2013), composed by three “root states”:
   1. $\mathbb{Z}_2$: Surface $\mathbb{Z}_2$ topo order. Both $e$ and $m$ carry half-charge; Witten effect with $\Theta = 2\pi \text{mod} 4\pi$ (so-called by $e_m e_m$ state) (in TI it is $\pi$ response, see seminal work Qi-Hughes-Zhang 2008)
   2. $\mathbb{Z}_2$: Surface $\mathbb{Z}_2$ topo order. Both $e$ and $m$ carry Kramer’s degeneracy. (so-called $e_T m_T$ state)
   3. $\mathbb{Z}_2$: Surface $\mathbb{Z}_2$ topo order. $e$, $m$, and $\epsilon$ are fermions. Beyond group cohomology. (so-called $e_f m_f$ state)
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   1. $\mathbb{Z}_2$: Surface $\mathbb{Z}_2$ topo order. Both $e$ and $m$ carry half-charge; Witten effect with $\Theta = 2\pi \mod 4\pi$ (so-called by $\text{ecmc}$ state) (in TI it is $\pi$ response, see seminal work Qi-Hughes-Zhang 2008)
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   3. $\mathbb{Z}_2$: Surface $\mathbb{Z}_2$ topo order. $e$, $m$, and $\epsilon$ are fermions. Beyond group cohomology. (so-called $\text{efmf}$ state)

3. However, their bulk field theory is, STILL, not well established yet.

4. **We aim to solve this problem and do more than that!!** (see: Ye, Gu 1410.2594)
“Derivation” of a field theory (hydrodynamic approach)

Following CS derivation in FQH, *hydrodynamic approach*. (early progress in FQH: Fradkin, Wen,.......See also G. Y. Cho’s slides today)

- **Key assumption**: (1) Gapped; (2) density & current fluctuation are dominant d.o.f. at low energies and long wavelengths.
Main results: All BTI obtained by condensing $2\pi$ vortex-lines in superfluid.

**Summary**

- **Trivial Mott Insulator** (presence of $b \wedge b$ term)
- **Superfluid (unusual symmetry transformation)**
- **BTI** (unusual symmetry transformation)
- **BTI** (presence of $b \wedge b$ term)

- **Right-corner**: one BTI beyond group cohomology. 
  \[ \sim \int K^{IJ} b^I \wedge da^J + \int \Lambda^{IJ} b^I \wedge b^J \]  
  (focused in this talk)

- **Left-corner**: two BTI within group cohomology. 
  \( \int K^{IJ} b^I \wedge da^J \). Either U(1) or $Z_2^T$ is transformed in an unusual way.
From Superfluid to trivial Mott insulator: Main steps

- **Duality**: Superfluid $\mathcal{L} = \frac{\rho}{2} (\partial_\mu \theta)^2 \leftrightarrow$ QED of 2-form gauge theory $b_{\mu\nu}$ coupled to vortex-lines $\Sigma_{\mu\nu}$.

$$\mathcal{L} = \frac{1}{48\pi^2 \rho} h_{\mu\nu\lambda} h_{\mu\nu\lambda} + \frac{i}{2} b_{\mu\nu} \Sigma_{\mu\nu},$$

where the field strength $h_{\mu\nu\lambda}$ is a rank-3 antisymmetric tensor:

$h_{\mu\nu\lambda} \overset{\text{def.}}{=} \partial_\mu b_{\nu\lambda} + \partial_\nu b_{\lambda\mu} + \partial_\lambda b_{\mu\nu}; \Sigma_{\mu\nu} \overset{\text{def.}}{=} \frac{1}{2\pi} \epsilon^{\mu\nu\lambda\rho} \partial_\lambda \partial_\rho \theta^\nu$
From Superfluid to trivial Mott insulator: Main steps

- **Duality**: Superfluid \( \mathcal{L} = \frac{\rho}{2} (\partial_{\mu} \theta)^2 \leftrightarrow \text{QED of 2-form gauge theory} \ b_{\mu\nu} \) coupled to vortex-lines \( \Sigma_{\mu\nu} \).

\[
\mathcal{L} = \frac{1}{48\pi^2 \rho} h_{\mu\nu\lambda} h_{\mu\nu\lambda} + \frac{i}{2} b_{\mu\nu} \Sigma_{\mu\nu},
\]
where the field strength \( h_{\mu\nu\lambda} \) is a rank-3 antisymmetric tensor:

\[
h_{\mu\nu\lambda} \overset{\text{def.}}{=} \partial_{\mu} b_{\nu\lambda} + \partial_{\nu} b_{\lambda\mu} + \partial_{\lambda} b_{\mu\nu}; \quad \Sigma_{\mu\nu} \overset{\text{def.}}{=} \frac{1}{2\pi} \epsilon^{\mu\nu\lambda\rho} \partial_{\lambda} \partial_{\rho} \theta^\nu
\]

- **Condensing** \( 2\pi \) vortex-line \( \Sigma_{\mu\nu} \) leads to a string condensate coupled to a *compact* 2-form gauge field \( b_{\mu\nu} \), i.e. a Higgs phase (techniques from string theory are involved, S. J. Rey 1989, M. Franz 2007):

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\mathcal{L} = \frac{1}{2} \phi_0^2 (\partial_{[\mu} \Theta_{\nu]} - b_{\mu\nu})^2 + \cdots
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• **Duality**: Higgs term $\longleftrightarrow b \wedge da$ term with non-compact 1-form U(1) gauge field $a_\mu$ and compact 2-form U(1) gauge field $b_{\mu\nu}$:

$$\mathcal{L} = i \frac{1}{4\pi} \epsilon^{\mu\nu\lambda\rho} b_{\mu\nu} \partial_{\lambda} a_\rho + ia_\mu j^{\mu}.$$

[1] $\frac{1}{4\pi}$ indicates GSD=1 on a 3-torus $T^3$.

[2] Symmetry transformation is also defined in a usual way. ($b$ transforms axial-like; $a$ transforms polar-like.)
From Superfluid to BTI (beyond group cohomology)

- Introduce a new quadratic term in the vortex-line condensate (induced by some interactions):

\[ \mathcal{L} \sim \frac{1}{2} \phi_0^2 \left( \partial_{[\mu} \Theta_{\nu]}^s - b_{\mu\nu} \right)^2 - i \frac{\Lambda}{16\pi} \epsilon^{\mu\nu\lambda\rho} \left( \partial_{[\mu} \Theta_{\nu]}^s - b_{\mu\nu} \right) \left( \partial_{[\lambda} \Theta_{\rho]}^s - b_{\lambda\rho} \right), \]
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• **Duality:** integrate out smooth phase \( \Theta_{\mu}^s \) and resolve constraint by introducing \( a_\mu \):

\[ \mathcal{L}_{\text{top}} = i \frac{1}{4\pi} \epsilon^{\mu\nu\lambda\rho} b_{\mu\nu} \partial_\lambda a_\rho + i \frac{\Lambda}{16\pi} \epsilon^{\mu\nu\lambda\rho} b_{\mu\nu} b_{\lambda\rho} \]
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- Gauge Invariance:

\[ b_{\mu\nu} \rightarrow b_{\mu\nu} + \partial_{[\mu} \xi_{\nu]} , \quad a_{\mu} \rightarrow a_{\mu} - \Lambda \xi_{\mu} . \]

The gauge transformation of \( a_{\mu} \) generalizes the usual one by including transverse components of \( \xi_{\mu} \).
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- **Physical meanings** of \( b \wedge b \): unlinking-linking process (collision)
From Superfluid to BTI (beyond group cohomology)

- Generalize to a multi-component theory:

\[ \mathcal{L}_{\text{top}} = i \frac{K^{IJ}}{4\pi} \epsilon_{\mu\nu\lambda\rho} b^I_{\mu\nu} \partial_\lambda a^J_\rho + i \frac{\Lambda^{IJ}}{16\pi} \epsilon_{\mu\nu\lambda\rho} b^I_{\mu\nu} b^J_{\lambda\rho}, \]

It is sufficient to consider symmetric matrix \( \Lambda^{IJ} \) and assume \( K^{IJ} \) to be an identity matrix of rank-\( N \), i.e.

\[ K = \text{diag}(1, 1, \cdots, 1)_{N \times N} = I \]

with \( I, J = 1, 2, \cdots, N \). \( \{a^I_\mu\} \) are non-compact 1-form U(1) gauge fields and \( \{b^I_{\mu\nu}\} \) are compact 2-form U(1) gauge fields, respectively.
From Superfluid to BTI (beyond group cohomology)

• **Generalize to a multi-component theory:**

\[
\mathcal{L}_{\text{top}} = i \frac{K^{IJ}}{4\pi} \epsilon^{\mu\nu\lambda\rho} b_{\mu\nu}^I \partial_\lambda a_\rho^J + i \frac{\Lambda^{IJ}}{16\pi} \epsilon^{\mu\nu\lambda\rho} b_{\mu\nu}^I b_{\lambda\rho}^J,
\]

It is sufficient to consider symmetric matrix \( \Lambda^{IJ} \) and assume \( K^{IJ} \) to be an identity matrix of rank-\( N \), i.e.

\[
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with \( I, J = 1, 2, \cdots, N \). \( \{a_\mu^I\} \) are non-compact 1-form U(1) gauge fields and \( \{b_{\mu\nu}^I\} \) are compact 2-form U(1) gauge fields, respectively.

• **Two independent \( \text{GL}(N, \mathbb{Z}) \) transformations:**

\[
b_{\mu\nu}^I = (W^{-1})^{IJ} b_{\mu\nu}^J, \quad a_\mu^I = (M^{-1})^{IJ} a_\mu^J,
\]

where, \( W, M \) are two \( N \) by \( N \) matrices with integer-valued entries and \( |\det W| = |\det M| = 1 \). A new set of parameters \((K, \Lambda, L_a, L_b)\) can be introduced via \((\therefore |\det K| \text{ is invariant})\):

\[
K = W^T K M, \quad \Lambda = W^T \Lambda W, \quad L_a = M^T L_a, \quad L_b = W^T L_b.
\]

Here, \( L_a \) and \( L_b \) are two qp vectors labelling excitations of particles and strings.
From Superfluid to BTI (beyond group cohomology)

In absence of time-reversal,

- **Flat-connection condition** of $b$ in an orientable manifold leads to $\Lambda^{IJ} \rightarrow \Lambda + 1$. No additional quantization condition. $\Lambda \in [0, 1)$, any nonzero $\Lambda$ can be continuously tuned to zero. back to trivial Mott insulator.

- **Conclusion**: $U(1)$ SPT in 3d is trivial, agreement to $H^4(U(1), U(1)) = \mathbb{Z}_1$. $\Lambda$ term has no effect.
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Imposing time-reversal in a usual way

- Definition of time-reversal in a usual way:

$$
\tau^a_i \tau^{-1} = \tau^a_i a^a_0, \quad \tau^b_{ij} \tau^{-1} = \tau^b_{ij} b^b_{0i}, \quad \tau^a_i \tau^{-1} = -\tau^a_i a^a_i, \quad \tau^b_{ij} \tau^{-1} = -\tau^b_{ij} b^b_{ij},
$$

$$
\tau^a = -\tau^b = \text{diag}(1, 1, \cdots, 1)_{N \times N}
$$

- Under the transformation, $\Lambda$ term becomes $-\Lambda$ term.

- **Flat-connection condition** of $b$ in a non-orientable manifold leads to: $\Lambda^{II} \rightarrow \Lambda^{II} + 4$ and $\Lambda^{IJ} \rightarrow \Lambda^{IJ} + 2$ ($I \neq J$)

∴ $\Lambda$-quantization induced by time-reversal: (compared to $U(1)$ SPT in 3d)

$$
\Lambda^{II} = 0, \pm 2, \Lambda^{IJ} = 0, \pm 1 \text{ (for } I \neq J). 
$$

A time-reversal protected quantization!
From Superfluid to BTI (beyond group cohomology)

- There are infinite # of $\Lambda$ matrices satisfying the condition.
- What is nontrivial $\Lambda$?
From Superfluid to BTI (beyond group cohomology)

- There are infinite # of $\Lambda$ matrices satisfying the condition.
- What is nontrivial $\Lambda$?
- Flat-connection condition leads to a ”CS” like surface theory with level $\Lambda$-matrix (“very abnormal” CS!! Gauss-Milgram sum is meaningless.)
- Two key statements toward ”BTI”:

**Definition**

**Physical observables of the surface theory** are composed by: ground state degeneracy, self-statistics and mutual statistics of gapped quasiparticles. chiral central charge $c^-$ is not an observable on the surface.

**Definition**

By “obstruction”, we mean that the set of physical observables cannot be reproduced on a 2d plane by any local bosonic lattice model with symmetry. Otherwise, the obstruction is free. Obstruction leads to nontrivial SPT.
From Superfluid to BTI (beyond group cohomology)

Under the time-reversal protected quantization condition
\( \Lambda'' = 0, \pm 2, \Lambda'^{ij} = 0, \pm 1 \) (for \( i \neq J \)):

§ All states with \( |\det \Lambda| = 1 \) are trivial (i.e. no obstruction):
- All are generated by two fundamental blocks up to \( \mathbb{GL} \) transformation:
  \[
  \Lambda_{t1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda_{t2} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.
  
  where, \( \Lambda_{t2} \) also doesn’t break TR since \( c^- = 8 \) is not an observable.
- **Stacking Recipe** for all such states \( c^- = 0, \pm 8, \pm 16, \cdots \): Apply \( \mp \Lambda_{t2} \) to remove \( c^- \).) Surface physical observables are unaffected!
From Superfluid to BTI (beyond group cohomology)

Under the time-reversal protected quantization condition
\( \Lambda'' = 0, \pm 2, \Lambda^{I J} = 0, \pm 1 \) (for \( I \neq J \)):

\[ \Lambda'' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda_t^2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix} \]

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\[ \Lambda'' = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad \Lambda''^T = -\Lambda \]

All states with \( |\det \Lambda| > 1 \) and \( W^T \Lambda W = -\Lambda, \exists W \in \mathbb{GL}(N, \mathbb{Z}) \) are also trivial. (such **GL** can be regularly defined on a 2d plane.)
From Superfluid to BTI (beyond group cohomology)

- Only possible obstruction can be realized by a modified GL transformation (realizable merely on surface):
  \[ W^T (\Lambda \oplus \Lambda_{t1} \oplus \Lambda_{t1} \cdots) W = (-\Lambda) \oplus \pm \Lambda_{t2} \oplus \pm \Lambda_{t2} \cdots, \exists W \in GL(N', \mathbb{Z}) \]
  which shifts chiral central charge, leaving surface physical observables unaffected.

- There is only one irreducible solution, namely, the Cartan matrix of SO(8) group (Wikipedia.org), denoted by \( \Lambda_{so8} \):
  \[ \Lambda_{so8} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}. \]

More precisely, the following transformation exists:

\[ W^T (\Lambda_{so8} \sum_{i=1}^{4} \oplus \Lambda_{t1}) W = (-\Lambda_{so8}) \oplus \Lambda_{t2}, \exists W \in GL(12, \mathbb{Z}). \]
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  More precisely, the following transformation exists:
  \[ W^T (\Lambda_{\text{so8}} \sum_{i=1}^{4} \oplus \Lambda_{t1}) W = (-\Lambda_{\text{so8}}) \oplus \Lambda_{t2}, \]
  \[ \exists W \in \text{GL}(12, \mathbb{Z}). \]

• A nontrivial SPT: \( e_{f} m_{f} \): Surface \( \mathbb{Z}_2 \) topo order. \( e, m, \) and \( \epsilon \) are fermions. Beyond group cohomology.
• Break TR symmetry if put on 2d plane (\textit{Stacking Recipe} cannot completely remove \( c^- \).)
Eventually arrive at the right-corner of phase diagram

- **Left-corner**: two BTI within group cohomology. \((b \wedge da)\).
  
  Either \(U(1)\) or \(Z_2^T\) is transformed in an unusual way.

- **Right-corner**: one BTI beyond group cohomology.
  
  \(\sim K^{ij} b^i \wedge da^j + \Lambda^{ij} b^i \wedge b^j, \Lambda = \Lambda_{SO(8)}\).
Last story: A new $\mathbb{Z}_2$ SPT beyond group cohomology

- $H^4(\mathbb{Z}_2, U(1)) = \mathbb{Z}_1$, all $\mathbb{Z}_2$ SPT states in 3d are trivial (group cohomology classification).
- **Our Conclusion:** If $bf + bb$ theory also describes a $\mathbb{Z}_2$ SPT state, we conclude: $\Lambda = 0 \mod 4$ is trivial while $\Lambda = 2 \mod 4$ is a nontrivial SPT:

  $$\mathcal{L} = i \frac{1}{4\pi} a_\mu \partial_\nu b_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho} + i \frac{\Lambda}{16\pi} b_{\mu\nu} b_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho}$$

- **Strategy:** by adding coupling terms with $B_{\mu\nu}$, gauging $\mathbb{Z}_2$ symmetry:

  $$\mathcal{L}_{\text{coupling}} = i \frac{1}{4\pi} B_{\mu\nu} \partial_\lambda a_\rho \epsilon^{\mu\nu\lambda\rho} + i \frac{2}{4\pi} B_{\mu\nu} \partial_\lambda A_\rho \epsilon^{\mu\nu\lambda\rho}$$

where, $A_\mu$ is used for higgsing $B_{\mu\nu}$ down to $\mathbb{Z}_2$. Then, integrating out SPT d.o.f., ends up with:

  $$S = i \int d^4x \frac{2}{4\pi} B_{\mu\nu} \partial_\lambda A_\rho \epsilon^{\mu\nu\lambda\rho} + i \int d^4x \frac{\Lambda}{16\pi} B_{\mu\nu} B_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho}$$

  $$= i \int d^4x \frac{2}{8\pi} B_{\mu\nu} F_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho} + i \int d^4x \frac{\Lambda}{16\pi} B_{\mu\nu} B_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho}$$

  $$\overset{\text{def.}}{=} i \frac{2}{2\pi} \int B \wedge dA + i \frac{\Lambda}{4\pi} \int B \wedge B.$$
Last story: A new $\mathbb{Z}_2$ SPT beyond group cohomology

- Flat-connection condition leads to: $\Lambda \rightarrow \Lambda + 2^2$
- Invariance under large gauge transformation leads to: $\Lambda/2 \in \mathbb{Z}$
- Thus, $\Lambda = 0$, or 2.
Last story: A new $\mathbb{Z}_2$ SPT beyond group cohomology

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- Thus, $\Lambda = 0$, or 2.

Thus, before gauging, a corresponding new $\mathbb{Z}_2$ SPT!

This probe theory is complete or consistent? an open question.
Summary and open questions

Summary in brief:

1. Projective construction leads to a large classes of 2d SPT ground states. arXiv:1212.2121; 1408.1676;

2. Bulk field theory of 3d bosonic topological insulator is constructed in a physical way. 1410.2594

3. Bulk definition is far more important than pure boundary argument.

Open questions:

1. Realistic spin lattice models that support the ground state from the method of “projective construction” (e.g. Mei, Wen, arXiv:1407.0869)

2. 3d field theory of generic SPT states, other symmetries.

3. A conclusive result of 3d field theory of fermion SPT (free and interacting), e.g. fermionic topological insulators. (previous efforts ever made by Cho, Moore, Ryu, · · · ) spin manifold?

Thank collaborators (X.-G. Wen, Z.-C. Gu, Z.-X. Liu, J.-W. Mei) and mentor S.-S. Lee in PI.

Thank PI for financial supports. Discussion with G. Baskaran about superfluid is acknowledged. Thanks for your attentions!