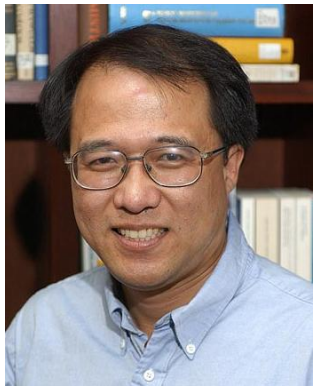
Symmetry-protected topological phases: Projective construction & Bulk field theory

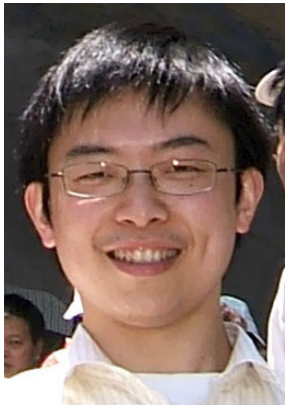
Péng Yè

SPT enthusiasts @ 3.14159

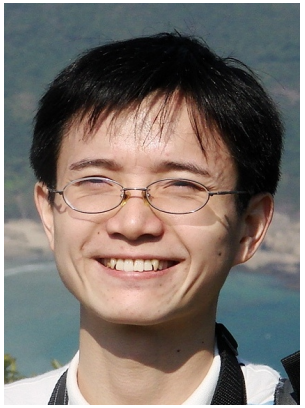
Collaboration with great colleagues, friends below:



Xiao-Gang Wen (MIT)



Zheng-Cheng Gu (PI)



Zheng-Xin Liu (Tsinghua)

Jia-Wei Mei (PI)

Focused references:

- Projective construction of 2d SPT: **arXiv:1212.2121, 1408.1676**
- Bulk field theory of 3d SPT: **arXiv:1410.2594**.

Related but not included here:

- Proj. constr. of 3d SPT: 1303.3572 [see my Princeton talk in Mar.];
- Response theory: 1306.3695;
- 1d SPT order of doped spin chain: 1310.6496;
- 3d SPT: solvable lattice model (Ye, Gu, in preparation)

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Defining triviality is quite nontrivial. $\longrightarrow$ Our development of physics is to continuously redefine triviality by deeper and deeper probes / insights.

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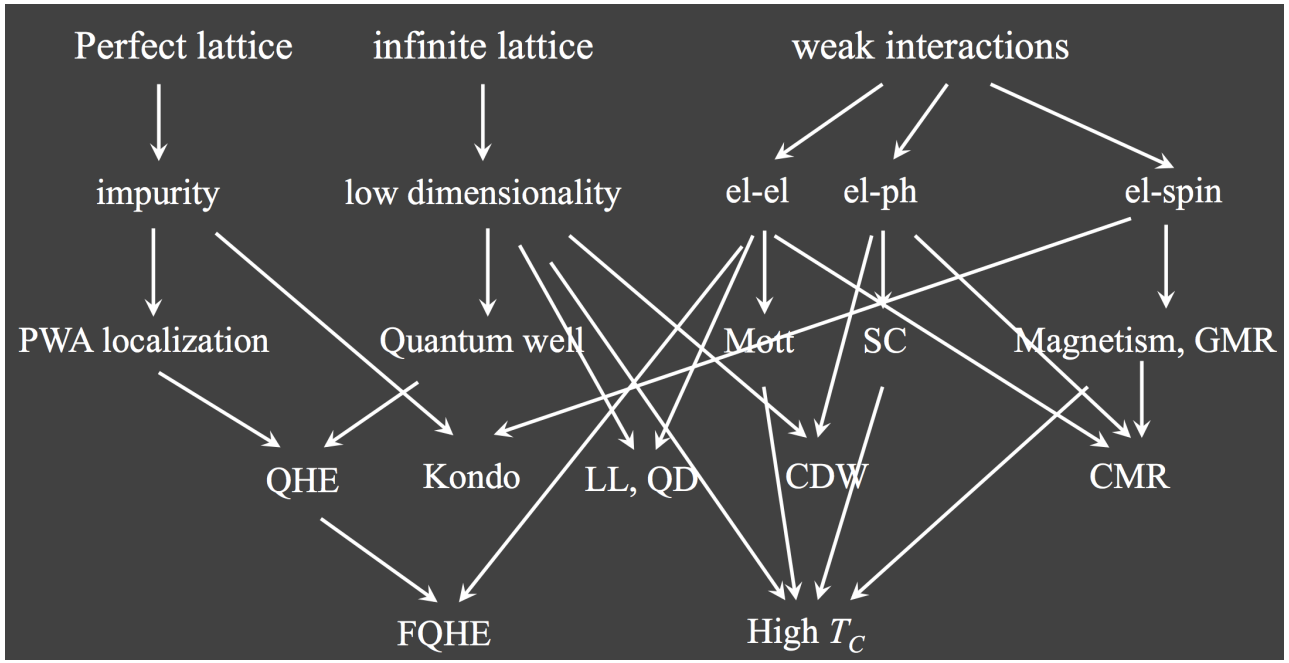
Indeed condensing topological defects generically drives a phase transition to exotic states of matter out from a conventional state.

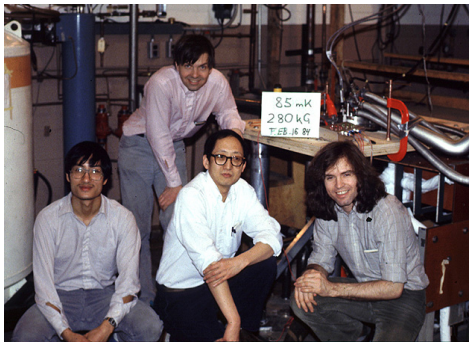
Contents

Section 1 General review

Hundred years' **hard** Condensed Matter (Solid State)

Physics in progress... Great triumph in 1980s.





Order

A central concept: "*order*" , a way of organization in many-body system.

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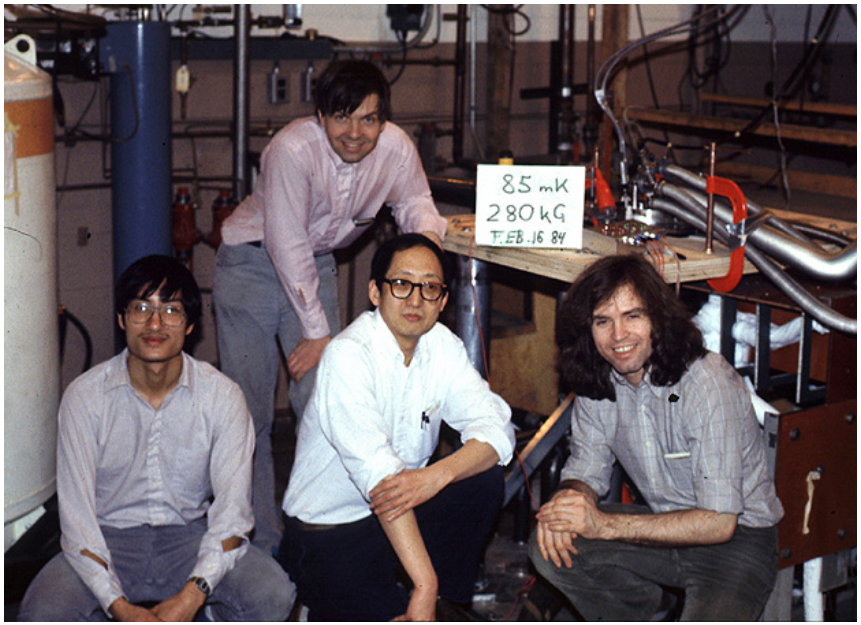
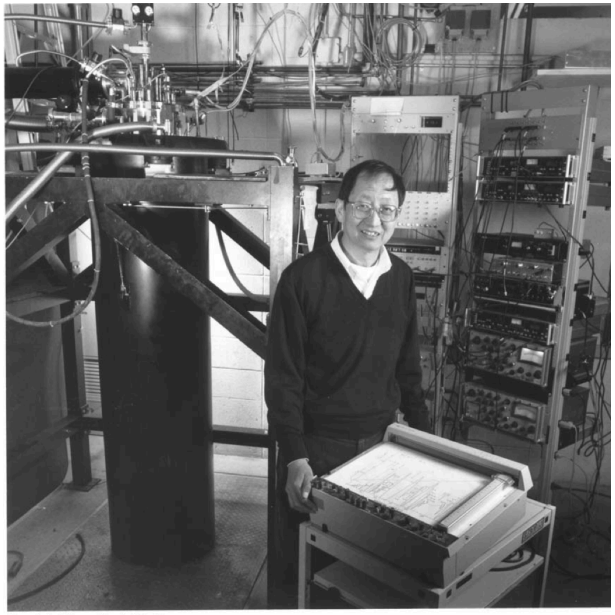
Symmetry-breaking order $G \longrightarrow H$

- Symmetry-breaking order (Landau theory) works very well for a long time.
- Local order parameter, as a function, is determined by G/H.

Fractional Quantum Hall states and topological order





Daniel Chee Tsui, Ph.D.
Arthur Legrand Doty Professor of Electrical Engineering
Princeton University (photo: Denise Applewhite)

Experimental conditions: **Clean** 2DEG ($\mu \sim 10^5 \mathrm{cm}^2/\mathrm{V{\cdot}s}$), **low** electron density ($n \sim 10^{11}/\mathrm{cm}^2$); **low** temperature, **high** magnetic field.
Unexpected results: Wigner crystal or CDW? in Prof. Tsui's seminal paper.
Now we know: **Topological order**— (Wen) non-symmetry-breaking order; **long-range entanglement**. (Gu, Wen, Chen, et.al.) $\longrightarrow$ quantum information, computing,...

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Some star examples in more realistic electronic materials: ("fermionic SPT")

1. Topological insulator (3d, time-reversal)
2. Quantum spin Hall insulator (2d TI, time-reversal)
3. Topological crystalline insulator. (space group, see, also, S. Ryu's talk)
4. .....

99% can be modelled by band theory, or, weak interacting but has a well-defined free-fermion counterpart.

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99% can be modelled by band theory, or, weak interacting but has a well-defined free-fermion counterpart.

Classification of free-fermion nontrivial SPT is well-known now (Ryu, Schnyder, Furusaki, Ludwig; Kitaev, $\cdots$)

Classification of interacting-fermion nontrivial SPT is under debate (Gu, Wen, Senthil, Kapustin, Ryu, Qi....many people) !!**A frontier of research**!!.

SPT in bosons/spin

In boson SPT, strong interaction and strong correlation effects are necessary!
Avoid boson condensation (boson system); avoid spin order (spin paramagnets)

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Some examples with concrete lattice models:

1. **Haldane phase** in spin-1 AKLT (or Heisenberg) is a bosonic SPT, a short-range entangled state. (Pollmann, et.al. Gu, Wen, et.al.)
2. **Topological Ising paramagnet** (Levin, Gu 2012). A nontrivial 2d SPT protected by Z_2 symmetry.
3. **Bosonic topological superconductor** (Burnell, Chen, et.al. 1302.7072). A nontrivial 3d SPT protected by TR sym.

Progress in SPT

Intensive study in North America (quite incomplete list. sorry for missing.)

- Group cohomology classification and fixed-point Lagrangian (Xie Chen, Gu, Liu, Wen)
- Classification based on Chern-Simons theory (Lu, Vishwananth)
- Classification based on non-linear σ field theory with θ-term (C. Xu's group)
- Classification based on "cobordism" (Kapustin, et.al.)

- SPT—gauge theory duality (Levin, Gu; Wen, et.al.)
- 3d bosonic TI (Vishwanath, Senthil, Metlitski, Ye, Wen, et.al.)
- 2d projective construction (Ye, Wen; Lu, Lee, et.al.)
- Boundary anomaly (S. Ryu's group; Wang, Wen, et.al.)
- 3d field theory (Xu, Senthil; Ye, Gu; ...)
- $\cdots$ See review article by T. Senthil arXiv: 1405.4015 and references therein.

Section 2 Projective construction of 2d SPT states

Projective construction in non-Abelian QH states

At least, two useful analytic ways to construct non-Abelian QH states:

1. Correlation functions in CFT: edge states (Blok, Wen, Wu, Nayak, Wilczek, Read, Rezayi, Cappelli, et.al.), topologically invariant, two different bdr (cut through plane or through time)

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2. **Projective (parton) construction**: both edge and bulk field theories. (Wen, Blok, Barkeshli, et.al.)

Proj. constr. of non-Abelian QH: Wen's recipe '99

1. **Introduce a few partons** ψ_a $(a = 1, \cdots, n)$, each carries Q_a charge:
$$\boxed{\mathcal{L}_{parton} = i\psi_a^\dagger \partial_t \psi_a + \frac{1}{2m}\psi_a^\dagger(\partial_i - iQ_a A_i)^2 \psi_a}$$
2. **Introduce an "electron" operator**: $\Psi_e = \sum_m C_m \prod_a \psi_a^{n_a^{(m)}}(z)$, where, $n_a^{(m)} = 0, 1$, and, $\sum_a Q_a n_a^{(m)} = e$.
3. $\psi_a \to W_{ab}\psi_b$ leaves Ψ_e and electron current operators J_e unchanged. Thus, $W^\dagger Q W = Q$, where, $Q = diag(Q_1, Q_2, \cdots)$. $W \in \mathcal{G}$ (gauge group).
4. Consider $\mathcal{G}_c \in \mathcal{G}$, a connected piece of $\mathcal{G}$, which contains identity. Thus, **define an emergent gauge field** $a_\mu \in \mathscr{L}_{\mathcal{G}_c}$, i.e. Lie algebra of $\mathcal{G}_c$. Thus, bulk theory becomes:
$$\boxed{\mathcal{L} = i\psi_a^\dagger[\delta_{ab}\partial_t - i(a_0)_{ab}]\psi_b + \frac{1}{2m}\psi_a^\dagger(\partial_i - iQA_i - ia_i)^2_{ab}\psi_b}$$
5. To ensure the **bulk is gapped**, one may consider partons form IQH states with filling $\nu_a = m_a$. **Integrating** partons leads to:
$$\boxed{\mathcal{S} = \frac{1}{4\pi}\mathrm{Tr}(M(a \wedge \mathrm{d}a + \frac{2}{3}a \wedge a \wedge a)) + \frac{\mathrm{Tr}(MQ^2)}{4\pi} A \wedge dA}$$
where, crossing term is eliminated by demanding: $\mathrm{Tr}(tMQ) = 0$, with arbitrary matrix $t \in \mathscr{L}_{\mathcal{G}_c}$. $M = diag(m_1, m_2, \cdots)$
6. Deriving **edge states** (Kac-Moody algebra) via coset theory or OPE.

Proj. constr. of SPT: Ye-Wen's program '12

To understand symmetry transformation on matter fields, we preserve partons d.o.f..

- Parton field current $\boxed{J^I_\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a^I_\lambda}$. Automatically resolve $\partial^\mu J^I_\mu = 0$. (a^I, at this step, is a non-compact, 1-form U(1) gauge connection, defined on the real axis.)

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- Projection is done by imposing a "**constraint**" on parton currents (i.e. fluxes) in path-integral measure:

$$\mathscr{F}(J^1_\mu, J^2_\mu, \cdots) = 0 \longrightarrow \mathscr{F}(a^1_\mu, a^2_\mu, \cdots) = 0$$

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To understand symmetry transformation on matter fields, we preserve partons d.o.f..

- Eventually, the **bulk theory** is described by a new $\tilde{K}$-Chern-Simons theory. (*assuming the constraint is a linear equation*)
  1. **If det**$\tilde{K} = 0$, then, apply $\mathbb{GL}$ transformation to separate the zero eigenvalue.
     1. *Spontaneous Symmetry-breaking order with Goldstone modes if zero eigenvalues carry symmetry charge. Monopole is suppressed. Maxwell term sets in.*
     2. *If not, zero eigenvalues are in Polyakov confined phase (monopole condensation). Directly Remove it!!*

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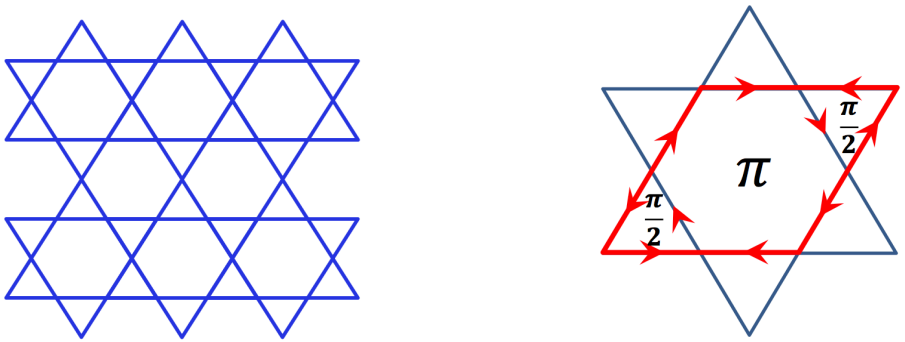
- **Edge reconstruction** (Sometimes, chiral edge $\longrightarrow$ non-chiral edge!!)

Proj. constr. of U(1) $\times$ U(1) SPT states

Start with a spin-1 system, $S = f^\dagger \Gamma f$, where Γ are three angular momentum generators. Partons: $f = (f_{+1}, f_0, f_{-1})$.

- Fill three Chern bands on a Kagome lattice (with Chern number C_1, C_0, C_{-1}).



- Minimal symmetry is: U(1)$\times$U(1) (conservation of both $\sum_i S_i^z$ and $\sum_i (S_i^z)^2$)
- Projection is done by imposing a constraint on gauge fluxes: $J_\mu^{+1} + J_\mu^0 + J_\mu^{-1} = K_\mu$, where, $K_\mu = (1, 0, 0)$.

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Consider $\mathbf{C_1} = \mathbf{1}, \mathbf{C_0} = -\mathbf{1}, \mathbf{C_{-1}} = \mathbf{1}$: The free parton theory is a "fermionic" Chern-Simons theory with chiral edge $(1 - 1 + 1 \neq 0)$

$$\mathcal{L} = \frac{1}{4\pi} \begin{pmatrix} a^1_\mu & a^0_\mu & a^{-1}_\mu \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \partial_\nu \begin{pmatrix} a^1_\lambda \\ a^0_\lambda \\ a^{-1}_\lambda \end{pmatrix} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} A^{S^z}_\mu \partial_\nu \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a^1_\lambda \\ a^0_\lambda \\ a^{-1}_\lambda \end{pmatrix} \epsilon^{\mu\nu\lambda}$$

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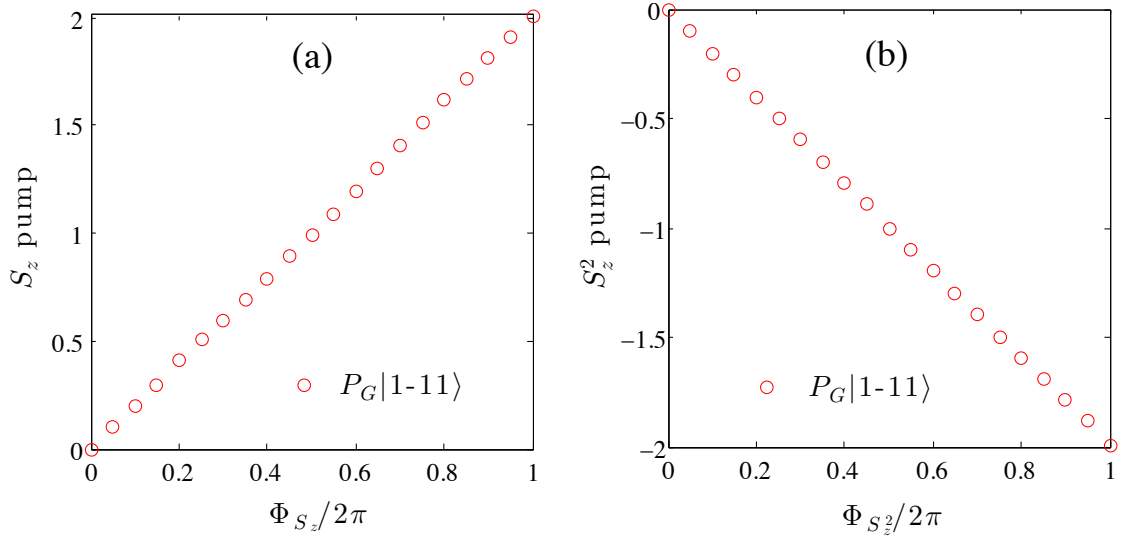
After Gutzwiller projection, $\mathrm{d}a^1 + \mathrm{d}a^{-1} + \mathrm{d}a^0 = 0$, we end up with:

$$\mathcal{L} = \frac{1}{4\pi} \begin{pmatrix} a^1_\mu & a^{-1}_\mu \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \partial_\nu \begin{pmatrix} a^1_\lambda \\ a^{-1}_\lambda \end{pmatrix} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} A^{S^z}_\mu \partial_\nu \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} a^1_\lambda \\ a^{-1}_\lambda \end{pmatrix} \epsilon^{\mu\nu\lambda}$$

[1] Groundstate wavefunction: $P_G|\text{free partons}\rangle$ (Monte Carlo testable)
[2] Edge is reconstructed to be non-chiral, with even-quantized nontrivial spin Hall conductance:

$$\sigma_H^{S^z} = \frac{1}{2\pi} \times \mathbf{2}.$$

Numerics: Even-quantized Hall conductance



One coin, two sides:

Non-zero even quantized Hall response $\longleftrightarrow$ All gapping terms on the boundary necessarily break symmetry $\longleftrightarrow$ nontrivial U(1) SPT order.

Gapping conditions

Understand more mathematically:

- In terms of bosonization, perturbation terms have the form of $\sim \cos(\ell^T \phi)$
- Is there a solution ℓ of equation $\ell^T K^{-1} \ell = 0$? (Haldane's null vector condition)
  - **No**. "intrinsic" state (chiral states and some nonchiral states (i.e. $\nu = 2/3$ FQH))
  - **Yes**. is there a solution ℓ further satisfying $\ell^T K^{-1} q = 0$?
    - **Yes**, trivial SPT. The perturbation term gaps edge without breaking symmetry.
    - **No**, nontrivial SPT. All gapping terms break U(1) symmetry. $\longrightarrow$ one can show: $q^T K^{-1} q$ is always even-integer (note that diagonal entries of K is even for boson systems and $|\det K| = 1$).
- **A side note:**

  # of symmetry-breaking perturbation $\gg$ # of gapping terms.

  For example, in-plane Zeeman term can not gap edge.
  $BS_x \sim \cos(2\phi_1 + \phi_{-1}) + \cos(2\phi_{-1} + \phi_1)$.

Numerical test

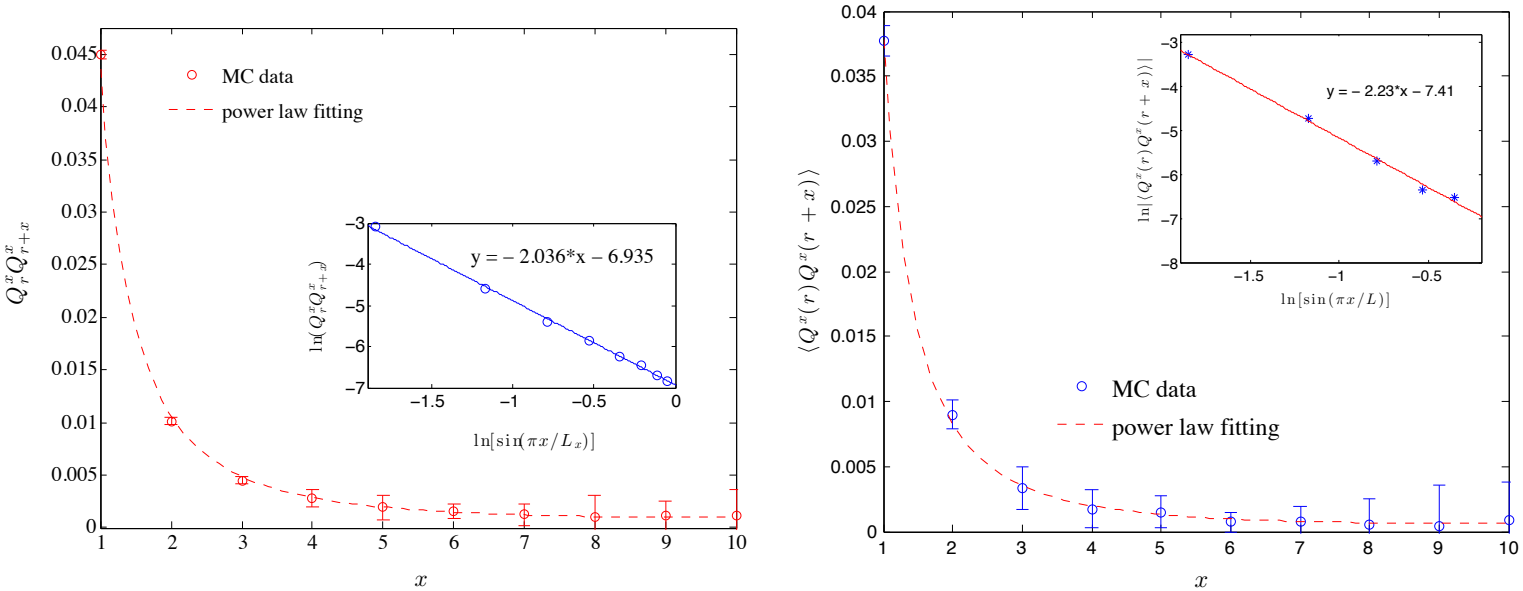


Figure : Correlation function. (1) power-law decay (2) still power-law decay after in-plane Zeeman is imposed. $B_x = 0.4$

Numerics: Gapped, and, trivial topological order

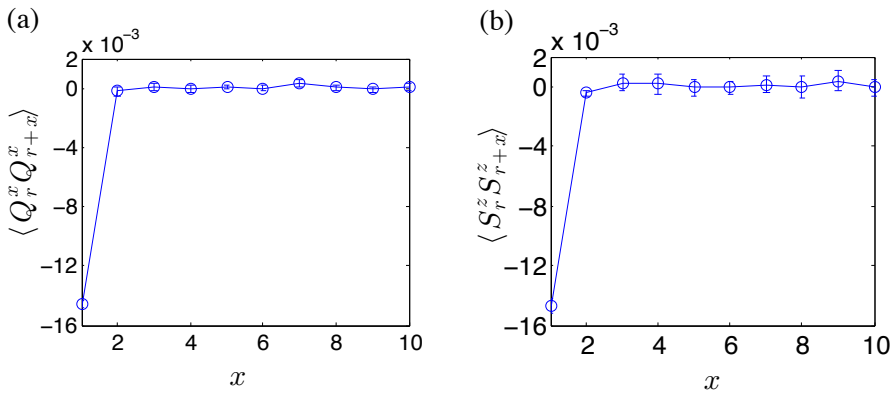


Figure : Bulk correlation function. "quadropole" $Q^x = (S^x)^2 - (S^y)^2$.

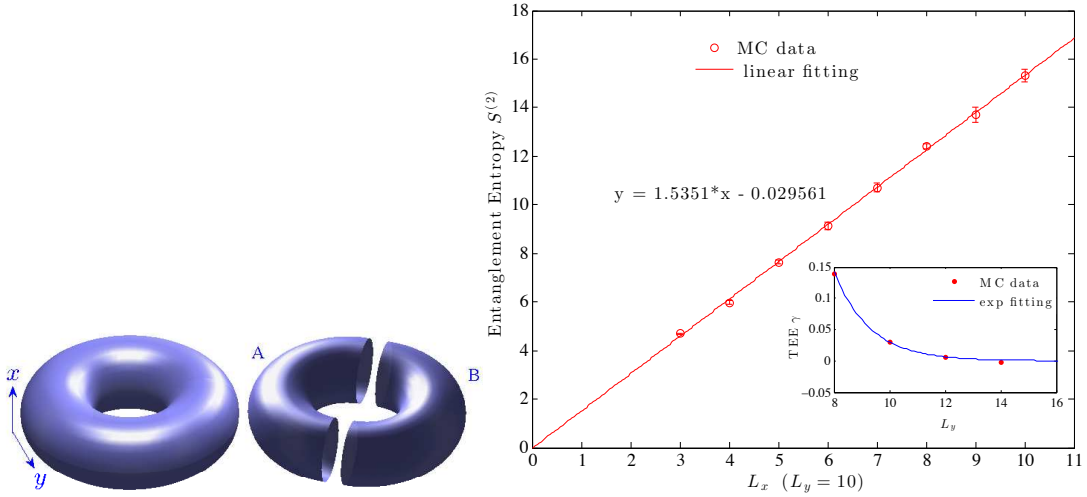


Figure : Topological entanglement entropy is zero.

Short summary of proj. constr. of SPT

What have been shown:

- Proj. Constr. provides ground state wave functions of SPT order with Abelian continuous symmetry, such as U(1)$\times$U(1). (Ye, Wen 1212.2121; Liu, Mei, Ye, Wen 1408.1676)

What have been not shown here:

1. Proj. Constr. can also construct SPT order with non-Abelian symmetry, such as SU(2), SO(3) (see Ye, Wen arXiv:1212.2121)
2. In 3d, techniques become complicated ($\because$ monopole/dyon might or might not be condensed in 3+1d gauge theory.) (see Ye, Wen arXiv:1303.3572)

Section 3 Bulk field theory of 3d SPT states

Physicists like surface

Totem and Motto of Surface Science

Figure : Physics is like a pizza, all the good stuff is on the surface! (pizza copied from Prof. Yayu Wang's lecture note)

However, bulk definition is more fundamental

- Bulk theory is a "**master equation**": e.g. CS theory of QH $\longrightarrow$ chiral Luttinger liquid.

However, bulk definition is more fundamental

- Bulk theory is a "**master equation**": e.g. CS theory of QH $\longrightarrow$ chiral Luttinger liquid.

- Two concerns on "surface-only research":
  - **Many-to-one correspondence** between surface and bulk is generically possible.
    e.g. QH at high Landau level (ν=8,12, see Cano,et.al. PRB 2014)
  - **Anomaly might not be fully canceled**.

Review of bosonic topological insulators (BTI)

Focus on BTI only.

Symmetry group $\mathrm{G} = \mathrm{U}(1) \rtimes \mathrm{Z}_2^T$ (Z_2^T: time-reversal symmetry, $\mathcal{T}^2 = 1$.)

1. Classification from group cohomology with U(1) coefficient:

$$\boxed{H^4(\mathrm{G}, \mathrm{U}(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2}$$ (Chen, Gu, Liu, Wen 2013)

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2. Recently, people found more: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, by surface argument (Vishwanath, Senthil, PRX2013), composed by three "root states":
   1. $\mathbb{Z}_2$: Surface Z_2 topo order. Both e and m carry half-charge; Witten effect with $\Theta = 2\pi \mathrm{mod} 4\pi$ (so-called by $\mathbf{e_C m_C}$ state) (in TI it is π response, see seminal work Qi-Hughes-Zhang 2008)
   2. $\mathbb{Z}_2$: Surface Z_2 topo order. Both e and m carry Kramer's degeneracy. (so-called $\mathbf{e_T m_T}$ state)
   3. $\mathbb{Z}_2$: Surface Z_2 topo order. e, m, and ϵ are fermions. Beyond group cohomology. (so-called $\mathbf{e_f m_f}$ state)

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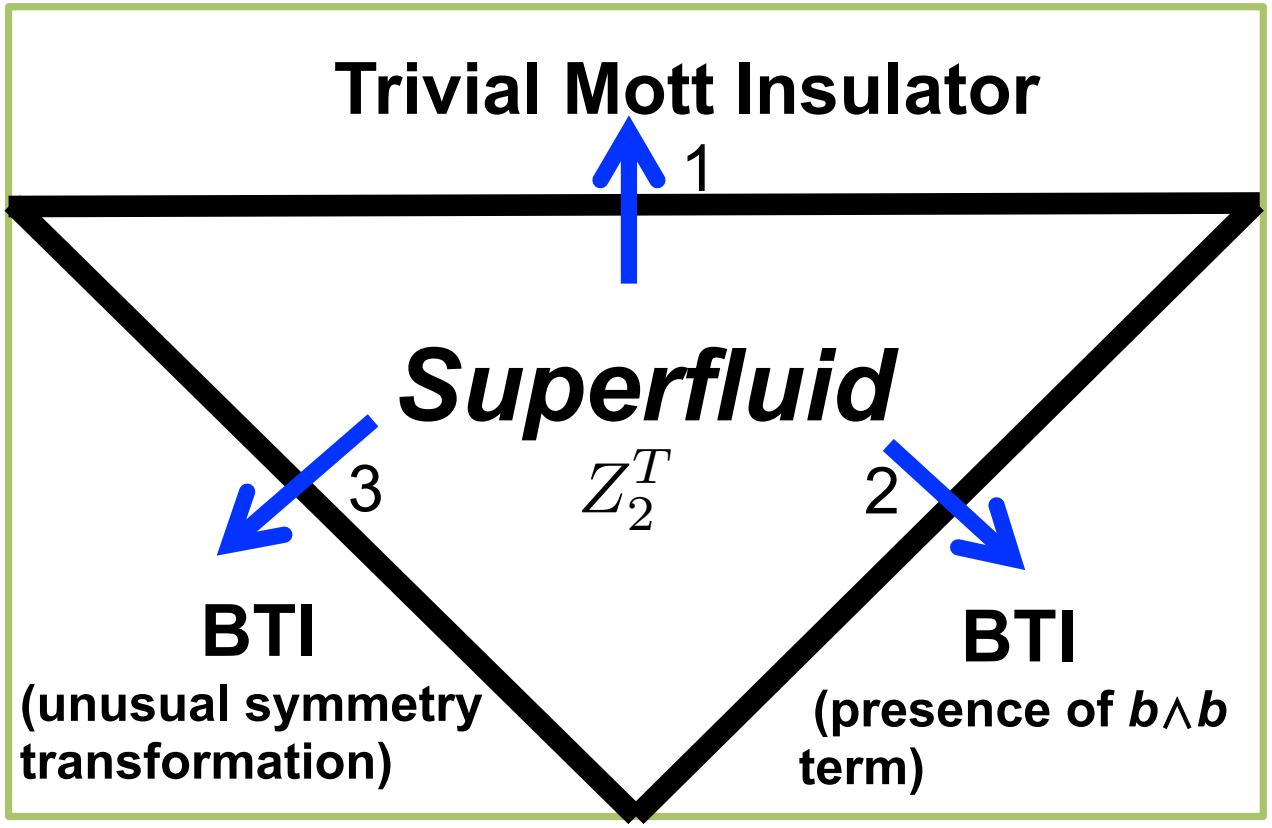
3. However, their bulk field theory is, STILL, not well established yet.
4. **We aim to solve this problem and do more than that!!** (see: Ye, Gu 1410.2594)

"Derivation" of a field theory (hydrodynamic approach)

Following CS derivation in FQH, *hydrodynamic approach*. (early progress in FQH: Fradkin, Wen,.....See also G. Y. Cho's slides today)

- **Key assumption**: (1) Gapped; (2) density & current fluctuation are dominant d.o.f. at low energies and long wavelengths.

Main results: All BTI obtained by condensing 2π vortex-lines in superfluid



- **Right-corner**: one BTI beyond group cohomology. ($\sim \int K^{IJ} b^I \wedge \mathrm{d}a^J + \int \Lambda^{IJ} b^I \wedge b^J$). (focused in this talk)
- **Left-corner**: two BTI within group cohomology. ($\int K^{IJ} b^I \wedge \mathrm{d}a^J$). Either U(1) or Z_2^T is transformed in an unusual way.

From Superfluid to trivial Mott insulator: Main steps

- **Duality**: Superfluid $\mathcal{L} = \frac{\rho}{2}(\partial_\mu \theta)^2 \longleftrightarrow$ QED of 2-form gauge theory $b_{\mu\nu}$ coupled to vortex-lines $\Sigma_{\mu\nu}$.

$$\mathcal{L} = \frac{1}{48\pi^2 \rho} h^{\mu\nu\lambda} h_{\mu\nu\lambda} + \frac{i}{2} b_{\mu\nu} \Sigma^{\mu\nu} ,$$

where the field strength $h_{\mu\nu\lambda}$ is a rank-3 antisymmetric tensor: $h_{\mu\nu\lambda} \overset{\text{def.}}{=\!=} \partial_\mu b_{\nu\lambda} + \partial_\nu b_{\lambda\mu} + \partial_\lambda b_{\mu\nu}$; $\Sigma^{\mu\nu} \overset{\text{def.}}{=\!=} \frac{1}{2\pi} \epsilon^{\mu\nu\lambda\rho} \partial_\lambda \partial_\rho \theta^{\mathrm{v}}$

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- **Condensing** 2π vortex-line $\Sigma_{\mu\nu}$ leads to a string condensate coupled to a *compact* 2-form gauge field $b_{\mu\nu}$, i.e. a Higgs phase (techniques from string theory are involved, S. J. Rey 1989, M. Franz 2007):

$$\mathcal{L} = \frac{1}{2} \phi_0^2 \left(\partial_{[\mu} \Theta_{\nu]} - b_{\mu\nu} \right)^2 + \cdots$$

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- **Duality**: Higgs term $\longleftrightarrow$ $b \wedge \mathrm{d}a$ term with non-compact 1-form U(1) gauge field a_μ and compact 2-form U(1) gauge field $b_{\mu\nu}$:

$$\mathcal{L} = i\frac{1}{4\pi} \epsilon^{\mu\nu\lambda\rho} b_{\mu\nu} \partial_\lambda a_\rho + i a_\mu j^\mu .$$

  [1] $\frac{1}{4\pi}$ indicates GSD=1 on a 3-torus $\mathbb{T}^3$.
  [2] Symmetry transformation is also defined in a usual way. (b transforms axial-like; a transforms polar-like.)

From Superfluid to BTI (beyond group cohomology)

- Introduce a new quadratic term in the vortex-line condensate (induced by some interactions):

$$\mathcal{L} \sim \frac{1}{2}\phi_0^2 \left(\partial_{[\mu}\Theta^s_{\nu]} - b_{\mu\nu}\right)^2 - i\frac{\Lambda}{16\pi}\epsilon^{\mu\nu\lambda\rho}\left(\partial_{[\mu}\Theta^s_{\nu]} - b_{\mu\nu}\right)\left(\partial_{[\lambda}\Theta^s_{\rho]} - b_{\lambda\rho}\right),$$

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$$b_{\mu\nu} \longrightarrow b_{\mu\nu} + \partial_{[\mu}\xi_{\nu]}\ ,\ a_\mu \longrightarrow a_\mu - \Lambda\xi_\mu\ .$$

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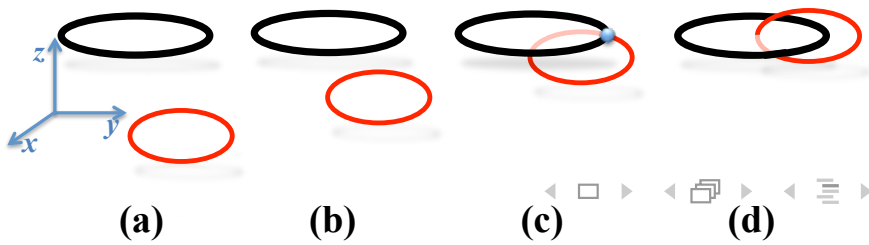
- **Physical meanings** of $b \wedge b$: unlinking-linking process (collision)



From Superfluid to BTI (beyond group cohomology)

- **Generalize to a multi-component theory**:

$$\mathcal{L}_{\text{top}} = i\frac{K^{IJ}}{4\pi}\epsilon^{\mu\nu\lambda\rho} b^I_{\mu\nu}\partial_\lambda a^J_\rho + i\frac{\Lambda^{IJ}}{16\pi}\epsilon^{\mu\nu\lambda\rho} b^I_{\mu\nu} b^J_{\lambda\rho},$$

It is sufficient to consider symmetric matrix Λ^{IJ} and assume K^{IJ} to be an identity matrix of rank-N, *i.e.*

$$K = \text{diag}(1, 1, \cdots, 1)_{N\times N} = \mathbb{I}$$

with $I, J = 1, 2, \cdots, N$. $\{a^I_\mu\}$ are non-compact 1-form U(1) gauge fields and $\{b^I_{\mu\nu}\}$ are compact 2-form U(1) gauge fields, respectively.

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- Two independent $\mathbb{GL}(N, \mathbb{Z})$ **transformations**:

$$\underline{b}^I_{\mu\nu} = (W^{-1})^{IJ} b^J_{\mu\nu}\ , \underline{a}^I_\mu = (M^{-1})^{IJ} a^J_\mu\ ,$$

where, W, M are two N by N matrices with integer-valued entries and $|\det W| = |\det M| = 1$. A new set of parameters $(\underline{K}, \underline{\Lambda}, \underline{\mathbf{L}}_a, \underline{\mathbf{L}}_b)$ can be introduced via ($\because$ $|\det K|$ is invariant):

$$\underline{K} = W^T K M\ , \underline{\Lambda} = W^T \Lambda W\ , \underline{\mathbf{L}}_a = M^T \mathbf{L}_a\ , \underline{\mathbf{L}}_b = W^T \mathbf{L}_b\ .$$

Here, $\mathbf{L}_a$ and $\mathbf{L}_b$ are two qp vectors labelling excitations of particles and strings.

From Superfluid to BTI (beyond group cohomology)

In absence of time-reversal,

- **Flat-connection condition** of b in an orientable manifold leads to $\Lambda^{IJ} \to \Lambda + 1$. No additional quantization condition. $\Lambda \in [0, 1)$, any nonzero Λ can be continuously tuned to zero. back to trivial Mott insulator.
- **Conclusion**: U(1) SPT in 3d is trivial, agreement to $\boxed{H^4(U(1), U(1)) = \mathbb{Z}_1}$. Λ term has no effect.

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Imposing time-reversal in a usual way

- Definition of time-reversal in a usual way:

$$\mathcal{T} a_0^I \mathcal{T}^{-1} = T^a_{IJ} a_0^J \ , \ \mathcal{T} b_{0i}^I \mathcal{T}^{-1} = T^b_{IJ} b_{0i}^J \ , \ \mathcal{T} a_i^I \mathcal{T}^{-1} = -T^a_{IJ} a_i^J \ , \ \mathcal{T} b_{ij}^I \mathcal{T}^{-1} = -T^b_{IJ} b_{ij}^J \ ,$$

$$T^a = -T^b = \mathrm{diag}(1, 1, \cdots, 1)_{N \times N}$$

- Under the transformation, Λ term becomes $-\Lambda$ term.
- **Flat-connection condition** of b in a non-orientable manifold leads to: $\Lambda^{II} \to \Lambda^{II} + 4$ and $\Lambda^{IJ} \to \Lambda^{IJ} + 2$ $(I \neq J)$
  $\therefore$ **Λ-quantization induced by time-reversal**: (compared to U(1) SPT in 3d)

$$\Lambda^{II} = 0, \pm 2 \, , \Lambda^{IJ} = 0, \pm 1 \ (\text{for } I \neq J) \, .$$

  A time-reversal protected quantization!

From Superfluid to BTI (beyond group cohomology)

- There are infinite # of Λ matrices satisfying the condition.
- What is nontrivial Λ?

From Superfluid to BTI (beyond group cohomology)

- There are infinite # of Λ matrices satisfying the condition.
- What is nontrivial Λ?
- Flat-connection condition leads to a "CS" like surface theory with level Λ-matrix ("very abnormal" CS!! Gauss-Milgram sum is meaningless.)
- Two key statements toward "BTI":

Definition

Physical observables of the surface theory *are composed by: ground state degeneracy, self-statistics and mutual statistics of gapped quasiparticles. chiral central charge c^- is not an observable on the surface.*

Definition

*By "**obstruction**", we mean that the set of physical observables cannot be reproduced on a 2d plane by any local bosonic lattice model with symmetry. Otherwise, the obstruction is free. Obstruction leads to nontrivial SPT.*

From Superfluid to BTI (beyond group cohomology)

Under the time-reversal protected quantization condition $\Lambda^{II} = 0, \pm 2\,, \Lambda^{IJ} = 0, \pm 1$ (for $I \neq J$):

§ All states with $|\det\Lambda| = 1$ are trivial (i.e. no obstruction):

- All are generated by two fundamental blocks up to $\mathbb{GL}$ transformation:

$$\Lambda_{\mathrm{t}1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda_{\mathrm{t}2} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

  where, $\Lambda_{\mathrm{t}2}$ also doesn't break TR since $c^- = 8$ is not an observable.

- *Stacking Recipe* for all such states $c^- = 0, \pm 8, \pm 16, \cdots$: Apply $\mp\Lambda_{\mathrm{t}2}$ to remove c^-.) Surface physical observables are unaffected!

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§ All states with $|\det\Lambda| > 1$ and $W^T \Lambda W = -\Lambda$, $\exists\, W \in \mathbb{GL}(N, \mathbb{Z})$ are also trivial. (such $\mathbb{GL}$ can be regularly defined on a 2d plane.)

From Superfluid to BTI (beyond group cohomology)

- Only possible obstruction can be realized by a modified GL transformation (realizable merely on surface):

  $$W^T(\Lambda \oplus \Lambda_{\mathrm{t1}} \oplus \Lambda_{\mathrm{t1}} \cdots) W = (-\Lambda) \oplus \pm\Lambda_{\mathrm{t2}} \oplus \pm\Lambda_{\mathrm{t2}} \cdots, \exists W \in \mathbb{GL}(N', \mathbb{Z})$$

  which shifts chiral central charge, leaving surface physical observables unaffected.

- There is only one irreducible solution, namely, the Cartan matrix of SO(8) group (Wikipedia.org), denoted by Λ_{so8}:

  $$\Lambda_{\mathrm{so8}} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

  More precisely, the following transformation exists:

  $$W^T(\Lambda_{\mathrm{so8}} \sum^{4} \oplus \Lambda_{\mathrm{t1}}) W = (-\Lambda_{\mathrm{so8}}) \oplus \Lambda_{\mathrm{t2}},$$
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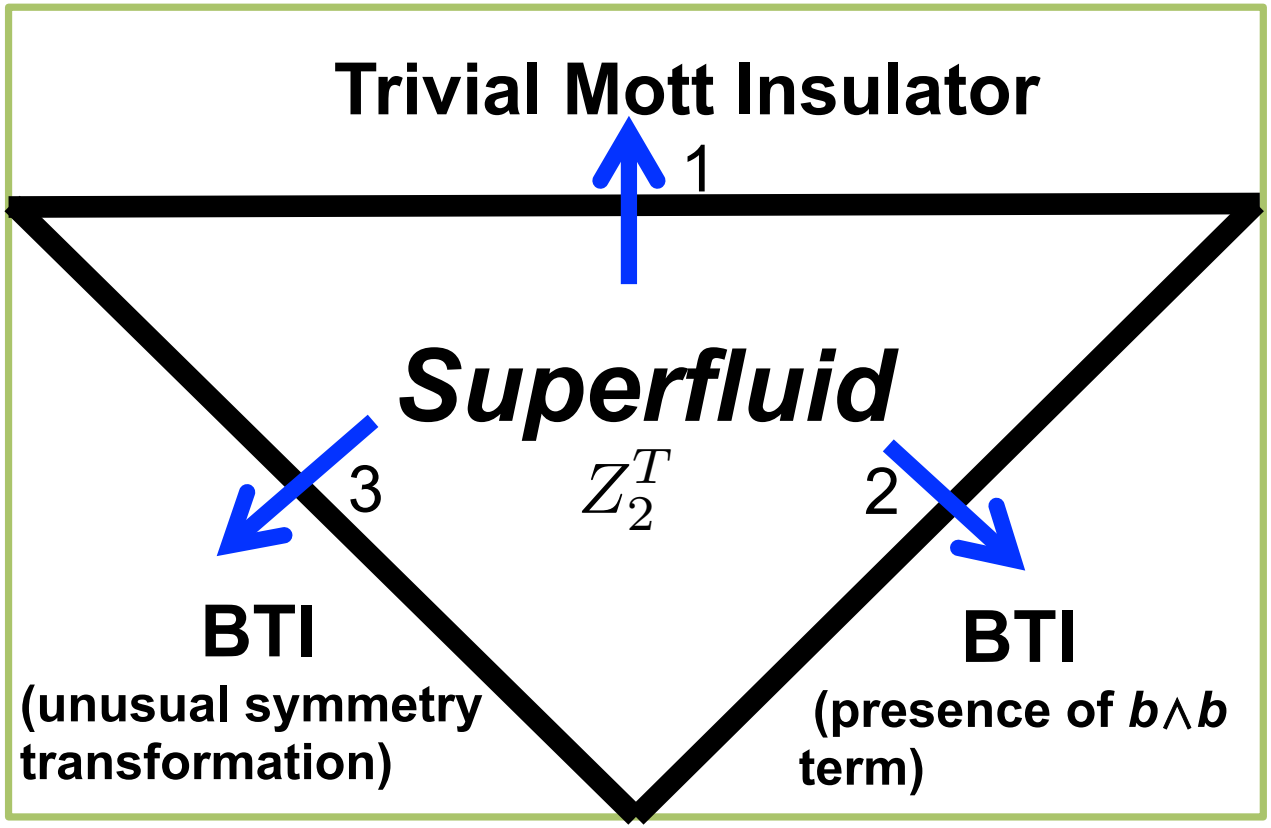
- A nontrivial SPT: $\mathbf{e_f m_f}$: Surface Z_2 topo order. e, m, and ϵ are fermions. Beyond group cohomology.
- Break TR symmetry if put on 2d plane (*Stacking Recipe* cannot completely remove c^-.)

Eventually arrive at the right-corner of phase diagram



- **Left-corner**: two BTI within group cohomology. ($b \wedge \mathrm{d}a$).
  Either U(1) or Z_2^T is transformed in an unusual way.
- **Right-corner**: one BTI beyond group cohomology.
  ($\sim K^{IJ} b^I \wedge \mathrm{d}a^J + \Lambda^{IJ} b^I \wedge b^J$, $\Lambda = \Lambda_{\mathrm{SO}(8)}$).

Last story: A new Z_2 SPT beyond group cohomology

- $H^4(Z_2, U(1)) = \mathbb{Z}_1$, all Z_2 SPT states in 3d are trivial (group cohomology classification).
- **Our Conclusion**: *If $bf + bb$ theory also describes a Z_2 SPT state, we conclude: $\Lambda = 0$ mod 4 is trivial while $\Lambda = 2$ mod 4 is a nontrivial SPT*:

$$\mathcal{L} = i\frac{1}{4\pi} a_\mu \partial_\nu b_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho} + i\frac{\Lambda}{16\pi} b_{\mu\nu} b_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho}$$

- **Strategy**: by adding coupling terms with $B_{\mu\nu}$, gauging Z_2 symmetry:

$$\boxed{\mathcal{L}_{coupling} = i\tfrac{1}{4\pi} B_{\mu\nu}\partial_\lambda a_\rho \epsilon^{\mu\nu\lambda\rho} + i\tfrac{2}{4\pi} B_{\mu\nu}\partial_\lambda A_\rho \epsilon^{\mu\nu\lambda\rho}}$$

where, A_μ is used for higgsing $B_{\mu\nu}$ down to Z_2. Then, integrating out SPT d.o.f., ends up with:

$$\begin{aligned} S =& i\int d^4x \frac{2}{4\pi} B_{\mu\nu}\partial_\lambda A_\rho \epsilon^{\mu\nu\lambda\rho} + i\int d^4x \frac{\Lambda}{16\pi} B_{\mu\nu} B_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho} \\ =& i\int d^4x \frac{2}{8\pi} B_{\mu\nu} F_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho} + i\int d^4x \frac{\Lambda}{16\pi} B_{\mu\nu} B_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho} \\ \overset{\text{def.}}{=\!=\!=}& i\frac{2}{2\pi}\int B\wedge \mathrm{d}A + i\frac{\Lambda}{4\pi}\int B\wedge B\,. \end{aligned}$$

Last story: A new Z_2 SPT beyond group cohomology

- Flat-connection condition leads to: $\Lambda \to \Lambda + 2^2$
- Invariance under large gauge transformation leads to: $\Lambda/2 \in \mathbb{Z}$
- Thus, $\Lambda = 0$, or 2.

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Thus, before gauging, a corresponding new Z_2 SPT !

This probe theory is complete or consistent? an open question.

Summary and open questions

§ Summary in brief:

1. Projective construction leads to a large classes of 2d SPT ground states. arXiv:1212.2121; 1408.1676;
2. Bulk field theory of 3d bosonic topological insulator is constructed in a physical way. 1410.2594
3. Bulk definition is far more important than pure boundary argument.

§ Open questions:

1. **Realistic spin lattice models** that support the ground state from the method of "projective construction" (e.g. Mei, Wen, arXiv:1407.0869)
2. 3d field theory of **generic SPT states, other symmetries**.
3. A conclusive result of 3d field theory of fermion SPT (free and interacting), e.g. **fermionic topological insulators**. (previous efforts ever made by Cho, Moore, Ryu, $\cdots$) spin manifold?

Thank collaborators (X.-G. Wen, Z.-C. Gu, Z.-X. Liu, J.-W. Mei) and mentor S.-S. Lee in PI. Thank PI for financial supports. Discussion with G. Baskaran about superfluid is acknowledged.Thanks for your attentions!